Annals of Physics 384 (2017) 85–104

Contents lists available at ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

The origin of the energy–momentum conservation law

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HIGHLIGHTS

- The action–reaction principle and energy–momentum conservation hold or fail together.
- Nontrivial topology is the reason for energy–momentum nonconservation in gravity.
- The total energy of a black hole is an ambiguous coordinatization-dependent quantity.
- This affair closely parallels that in the paradoxical Banach–Tarski theorem.

ARTICLE INFO

Article history:
Received 1 May 2017
Accepted 27 June 2017
Available online 15 July 2017

Keywords:
Action–reaction
Translation invariance
Energy and momentum conservation
Rearrangement of initial degrees of freedom

ABSTRACT

The interplay between the action–reaction principle and the energy–momentum conservation law is revealed by the examples of the Maxwell–Lorentz and Yang–Mills–Wong theories, and general relativity. These two statements are shown to be equivalent in the sense that both hold or fail together. Their mutual agreement is demonstrated most clearly in the self-interaction problem by taking account of the rearrangement of degrees of freedom appearing in the action of the Maxwell–Lorentz and Yang–Mills–Wong theories. The failure of energy–momentum conservation in general relativity is attributed to the fact that this theory allows solutions having nontrivial topologies. The total energy and momentum of a system with nontrivial topological content prove to be ambiguous, coordinatization-dependent quantities. For example, the energy of a Schwarzschild black hole may take any positive value greater than, or equal to, the mass of the body whose collapse is responsible for forming this black hole. We draw the analogy to the

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http://dx.doi.org/10.1016/j.aop.2017.06.018
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paradoxial Banach–Tarski theorem; the measure becomes a poorly defined concept if initial three-dimensional bounded sets are rearranged in topologically nontrivial ways through the action of free non-Abelian isometry groups.

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1. Introduction

By summing the basic advances in physics of the 19th century, Max Planck placed strong emphasis on the action–reaction principle as the rationale of momentum conservation [1]. On the other hand, following Noether's first theorem [2], we recognize that any dynamical system exhibits momentum conservation if the action of this system is invariant under space translations, in other words, the momentum conservation law stems from homogeneity of space.

In nonrelativistic mechanics, Newton's third law is consistent with the requirement of translation invariance. Indeed, the forces exerted on particles in an isolated two-particle system are on the same line, equal, and oppositely directed when the potential energy assumes the form $U(\mathbf{z}_1 - \mathbf{z}_2)$, where $\mathbf{z}_1$ and $\mathbf{z}_2$ are coordinates of these particles. However, this law is no longer valid in relativistic mechanics where the influence of one particle on another propagates at a finite speed, and the response arises with some retardation. Furthermore, energy and momentum are fused into energy–momentum whose conservation is attributed to homogeneity of Minkowski spacetime. So the Planck's insight into the reason for momentum conservation is gradually fading from the collective consciousness of theoretical physics.

Meanwhile there is one exceptional case, namely contact interactions, in which one particle acts on another and experiences back reaction at the same point, as exemplified by collisions and decays of pointlike particles. This form of relativistic interactions respects both Newton's third law and energy–momentum conservation, suggesting to consider the action–reaction principle in a broader sense and extend it to cover local interactions in classical field theories. The most familiar example can be found in the Maxwell–Lorentz electrodynamics in which the role of the electric charge e is twofold: e acts as both coupling between the point particle carrying this charge and electromagnetic field and the strength of the delta-function source of electromagnetic field.

To gain a clear view of whether the action–reaction principle has a direct bearing on energy–momentum conservation, one should invoke the self-interaction problem. This issue is studied in Sections 2 and 3 by the examples of Maxwell–Lorentz electrodynamics and Yang–Mills–Wong theory.

Turning to general relativity in Section 4, we conclude that both action–reaction principle and energy–momentum conservation cease to be true. The absence of energy–momentum conservation from this theory is due to the fact that the equation of gravitational field allows solutions which represent spacetime manifolds with nontrivial topology. Energy and momentum may thus become poorly defined concepts in general relativity. It transpires that the total energy of a Schwarzschild black hole may take any positive value greater than, or equal to, the mass of the collapsed body in different coordinatizations. The situation closely resembles that in the paradoxial Banach–Tarski theorem. We sketch the broad outline of this theorem and its potential relevance to the problem of poorly defined measure for total energy and momentum in Sections 4 and 5. The rearrangement of degrees of freedom appearing in the action and its role in facilitating the integral quantities to become well-defined is discussed in Section 5.

We follow the notation used in [3]. In Sections 2, 3 and 5, in which our attention is restricted essentially to the picture in Minkowski spacetime, we adopt the mainly negative signature $(+---)$ convenient to the description of world lines. In Section 4, we proceed from the idea of pseudo-Riemannian spacetime, and use the mainly positive signature $(-+++)$, which is particularly adapted to the description of 3-dimensional surfaces. We put the speed of light equal to unity throughout.

2. The Maxwell–Lorentz electrodynamics

The action

$$S = -\int d^4x \left(\frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu} + j^\mu A_\mu \right) - m_0 \int d\tau \, \sqrt{\dot{z}^\mu \dot{z}_\mu} \tag{1}$$

encodes the dynamics of a single charged particle interacting with electromagnetic field. Here,

$$j^\mu(x) = e \int_{-\infty}^{\infty} d\tau \, \dot{z}^\mu(\tau) \, \delta^4[x - z(\tau)] \tag{2}$$

is the current density produced by the particle moving along a smooth timelike world line $z^\mu(\tau)$ and carrying the charge e, and m_0 is the mechanical mass of this bare particle.

A closed system of this kind enjoys the property of translational invariance which affords energy–momentum conservation through the famous Noether argument.

The comparison of the source, Eq. (2), in the field equation

$$\mathcal{E}_\mu = \partial^\nu F_{\mu\nu} + 4\pi j_\mu = 0 \tag{3}$$

with the Lorentz force in the equation of motion for this charged particle

$$\varepsilon^\lambda = m_0 \ddot{z}^\lambda - e\dot{z}_\mu F^{\lambda\mu} = 0, \tag{4}$$

where the dot stands for the derivative with respect to the proper time s of the particle, shows that both are scaled by the same parameter e. This fact is consistent with the action–reaction principle: e measures both variation of the particle state for a given field state and variation of the field state for a given particle state.

Does this statement bear on energy–momentum conservation? To answer this question, we turn to the *self-interaction* problem. Naively, this problem is about interfacing the bare particle and electromagnetic field on the world line, which will hopefully reveal local energy–momentum balance of this contact interaction. We are therefore to address a simultaneous solution of Eqs. (3) and (4). To see this, consider the Noether identity

$$\partial_\mu T^{\lambda\mu} = \frac{1}{4\pi} \mathcal{E}_\mu F^{\lambda\mu} + \int_{-\infty}^{\infty} ds \, \varepsilon^\lambda(z) \, \delta^4 \left[x - z(s) \right], \tag{5}$$

where $T^{\mu\nu}$ is the total metric stress–energy tensor of this system,

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = \Theta^{\mu\nu} + t^{\mu\nu}, \tag{6}$$

$$\Theta^{\mu\nu} = \frac{1}{4\pi} \left(F^{\mu\alpha}F^\nu_\alpha + \frac{1}{4} \eta^{\mu\nu} F_{\alpha\beta}F^{\alpha\beta} \right), \tag{7}$$

$$t^{\mu\nu} = m_0 \int_{-\infty}^{\infty} ds \, \dot{z}^\mu(s) \dot{z}^\nu(s) \, \delta^4 \left[x - z(s) \right], \tag{8}$$

and $\mathcal{E}_\mu$ and ε^λ are, respectively, the left-hand sides of Eqs. (3) and (4). It follows from (5) that $\mathcal{E}_\mu = 0$ and $\varepsilon^\lambda = 0$ imply $\partial_\mu T^{\lambda\mu} = 0$, that is, the equation of motion for a bare particle (4), in which an appropriate solution to the field equation (3) has been used, is equivalent to the local conservation law for the total stress–energy tensor.

Imposing the retarded boundary condition, we obtain a solution to Eq. (3) in the Liénard–Wiechert form,

$$F^{\mu\nu}_{\text{ret}} = \frac{e}{\rho^2} \left(R^\mu V^\nu - R^\nu V^\mu \right), \tag{9}$$

$$V^\mu = \left[1 - (R \cdot \ddot{z}) \right] \frac{\dot{z}^\mu}{\rho} + \ddot{z}^\mu, \tag{10}$$

$R^\mu = x^\mu - z^\mu(s_{\rm ret})$ is a lightlike vector drawn from a point $z^\mu(s_{\rm ret})$ on the world line, where the signal was emitted, to the point x^μ, where the signal was received, and $\rho = R \cdot \dot{z}$ is the spatial distance between x^μ and $z^\mu(s_{\rm ret})$ in the instantaneously comoving Lorentz frame in which the charge is at rest at the retarded instant $s_{\rm ret}$. The field (9)–(10) is singular on the world line. Substituting it into (4) results in a divergent expression. This divergence is a manifestation of infinite self-interaction: the charged bare particle experiences its own electromagnetic field which is infinite at the point of origin.

A possible cure for this difficulty is to regularize the Liénard–Wiechert field $F^{\mu\nu}_{\rm ret}$ in a small vicinity of the world line. Take, for example, the field as a function of two variables $F^{\mu\nu}_{\rm ret}(x; z(s_{\rm ret}))$ and continue it analytically from null intervals between the observation points x^μ and the retarded points $z^\mu(s_{\rm ret})$ to timelike intervals that result from assigning $x^\mu = z^\mu(s_{\rm ret} + \epsilon)$ and keeping the second variable $z^\mu(s_{\rm ret})$ fixed [4]. A crucial step in removing the regularization is to change m_0 to a function of regularization, $m_0(\epsilon)$, add it to the divergent term $e^2/2\epsilon$, and assume that

$$m = \lim_{\epsilon \to 0} \left[m_0(\epsilon) + \frac{e^2}{2\epsilon} \right] \tag{11}$$

is finite and positive. Then the divergence disappears, and we arrive at the Lorentz–Dirac equation [5]

$$A^\mu = m\ddot{z}^\mu - \frac{2}{3}\, e^2 \left(\dddot{z}^\mu + \dot{z}^\mu \ddot{z}^2 \right) - f^\mu_{\rm ext} = 0, \tag{12}$$

where $f^\mu_{\rm ext} = e\dot{z}_\nu F^{\mu\nu}_{\rm ext}$ is an external four-force, with $F^{\mu\nu}_{\rm ext}$ being a free electromagnetic field.

Is it possible to regard (12) as the desired equation of local energy–momentum balance? Based on the wide-spread belief that the Abraham term

$$\Gamma^\mu = \frac{2}{3}\, e^2 \left(\dddot{z}^\mu + \dot{z}^\mu \ddot{z}^2 \right) \tag{13}$$

is the radiation reaction *four-force*, one would give a negative answer to this question. This is because the radiating particle feels a recoil equal to the negative of the Larmor emission rate

$$\dot{\mathcal{P}}^\mu = -\frac{2}{3}\, e^2\, \dot{z}^\mu \ddot{z}^2. \tag{14}$$

However, $-\dot{\mathcal{P}}^\mu$ cannot be considered as a four-force because it is not orthogonal to $\dot{z}^\mu$. On the other hand, Γ^μ is orthogonal to $\dot{z}^\mu$, but it differs from the anticipated recoil by the so-called Schott term $\frac{2}{3}\, e^2\, \dddot{z}^\mu$. Although the energy stored in the Schott term can be attributed to a reversible form of emission and absorption of field energy [5], its actual role appears mysterious.

Furthermore, the general solution to Eq. (12) with $f^\mu_{\rm ext} = 0$ is

$$\dot{z}^\mu(s) = e^\mu_0 \cosh(\alpha_0 + w_0 \tau_0\, e^{s/\tau_0}) + e^\mu_1 \sinh(\alpha_0 + w_0 \tau_0\, e^{s/\tau_0}), \tag{15}$$

where e^μ_0 and e^μ_1 are constant vectors such that $e_0 \cdot e_1 = 0$, $e_0^2 = -e_1^2 = 1$, $\tau_0 = 2e^2/3m$, α_0 and w_0 are arbitrary constants. The solution (15) is an embarrassing feature of the Lorentz–Dirac equation: a free charged particle moving along this world line continually accelerates,

$$\ddot{z}^2(s) = -w_0^2 \exp\left(2s/\tau_0\right), \tag{16}$$

and continually radiates. This *self-acceleration* seems *contrary to the energy–momentum conservation law* even though this law is assured by *translational invariance* of the action.

These paradoxial results[1] signal that self-interaction is a subtle issue whose treatment requires further refinements of the conceptual basis. A plausible assumption is that the extremization of the action, subject to the retarded condition, may result in unstable modes, which culminates in *rearranging* the initial degrees of freedom [3]. The action (1) is expressed in terms of mechanical variables $z^\mu(\tau)$ describing world lines of a bare charged particle and the electromagnetic vector

[1] For other paradoxes related to self-interaction in the Maxwell–Lorentz theory see, e. g., [6].

potential $A^\mu(x)$. The rearrangement of these degrees of freedom yields new dynamically independent entities, a *dressed* charged particle and *radiation*.

We begin with the local conservation law for the total stress–energy tensor

$$\partial_\lambda T^{\lambda\mu} = 0. \tag{17}$$

Recall that taking the local conservation law (17), as the starting point in the self-energy analysis, is as good as that of simultaneous solution of dynamical equations (3) and (4). Substituting the general solution of the field equation (3), $F^{\mu\nu} = F^{\mu\nu}_{\rm ret} + F^{\mu\nu}_{\rm ext}$, into (7) gives

$$\Theta^{\mu\nu} = -\frac{e^2}{4\pi\rho^4}\left[V^2R^\mu R^\nu - (R^\mu V^\nu + R^\nu V^\mu) + \frac{1}{2}\,\eta^{\mu\nu}\right] + \Theta^{\mu\nu}_{\rm mix}, \tag{18}$$

where the first term results from the self-field (9)–(10), and the second term contains mixed contributions of the self-field and free field. The first term splits into two parts $\Theta^{\mu\nu}_{\rm bound} + \Theta^{\mu\nu}_{\rm rad}$, where

$$\Theta^{\mu\nu}_{\rm bound} = -\frac{e^2}{4\pi\rho^4}\left[\frac{R^\mu R^\nu}{\rho^2}\,(1 - 2R\cdot\ddot{z}) - (R^\mu V^\nu + R^\nu V^\mu) + \frac{1}{2}\,\eta^{\mu\nu}\right], \tag{19}$$

$$\Theta^{\mu\nu}_{\rm rad} = -\frac{e^2}{4\pi\rho^4}\left[\ddot{z}^2 + \frac{1}{\rho^2}(\ddot{z}\cdot R)^2\right]R^\mu R^\nu. \tag{20}$$

The following local conservation laws hold off the world line [7]:

$$\partial_\mu\Theta^{\mu\nu}_{\rm bound} = 0, \quad \partial_\mu\Theta^{\mu\nu}_{\rm rad} = 0, \quad \partial_\mu\Theta^{\mu\nu}_{\rm mix} = 0. \tag{21}$$

A natural interpretation of (21) is that $\Theta^{\mu\nu}_{\rm bound}$, $\Theta^{\mu\nu}_{\rm rad}$, and $\Theta^{\mu\nu}_{\rm mix}$ are *dynamically independent* outside the world line [7]. There is no other decomposition of $\Theta^{\mu\nu}$ into parts which may be recognized as dynamically independent.

Since $\Theta^{\mu\nu}_{\rm rad}$ and $\Theta^{\mu\nu}_{\rm mix}$ behave like ρ^{-2} near the world line, they are integrable over a three-dimensional spacelike surface Σ, and, in view of (21), the surface of integration may be deformed from Σ to more geometrically motivated surfaces. It is convenient to substitute Σ by a tube T_ϵ of infinitesimal radius ϵ enclosing the world line to obtain

$$\mathcal{P}^\mu = \int_\Sigma d\sigma_\lambda\,\Theta^{\lambda\mu}_{\rm rad} = \lim_{\epsilon\to 0}\int_{T_\epsilon} d\sigma_\lambda\,\Theta^{\lambda\mu}_{\rm rad} = -\frac{2}{3}\,e^2\int_{-\infty}^{s} d\tau\,\ddot{z}^2(\tau)\dot{z}^\mu(\tau) \tag{22}$$

and

$$\wp^\mu = \int_\Sigma d\sigma_\lambda\,\Theta^{\lambda\mu}_{\rm mix} = \lim_{\epsilon\to 0}\int_{T_\epsilon} d\sigma_\lambda\,\Theta^{\lambda\mu}_{\rm mix} = -e\int_{-\infty}^{s} d\tau\,F^{\mu\nu}_{\rm ext}(z)\,\dot{z}_\nu(\tau). \tag{23}$$

$\mathcal{P}^\mu$ represents the four-momentum radiated by the charge e during the whole past history prior to the instant s. Indeed: (i) $\Theta^{\mu\nu}_{\rm rad}$ is a dynamically independent part of $\Theta^{\mu\nu}$; (ii) $\Theta^{\mu\nu}_{\rm rad}$ moves away from the charged particle with the speed of light, more precisely, $\Theta^{\mu\nu}_{\rm rad}$ propagates along the future light cone C_+ drawn from the emission point, $\Theta^{\mu\nu}_{\rm rad}R_\nu = 0$; (iii) the flux of $\Theta^{\mu\nu}_{\rm rad}$ goes as ρ^{-2} implying that the same amount of energy–momentum flows through spheres of different radii. Differentiating (22) with respect to s gives the Larmor four-momentum emitted by the accelerated charge per unit proper time, Eq. (14).

As for $\wp^\mu$, it is the four-momentum extracted from the free field $F^{\mu\nu}_{\rm ext}(x)$ during the whole past history up to the instant s.

By contrast, $\Theta^{\mu\nu}_{\rm bound}$ contains singularities ρ^{-3} and ρ^{-4} which are not integrable. Hence, an appropriate regularization is necessary. For example, employing a Lorentz-invariant cutoff prescription [3], one finds

$$P^\mu_{\rm bound} = {\rm Reg}_\epsilon\int_\Sigma d\sigma_\lambda\,\Theta^{\lambda\mu}_{\rm bound} = \frac{e^2}{2\epsilon}\,\dot{z}^\mu - \frac{2}{3}\,e^2\ddot{z}^\mu, \tag{24}$$

where ϵ is the cutoff parameter which must go to zero in the end of calculations. Since the flux of $\Theta^{\mu\nu}_{\rm bound}$ through C_+ is nonzero, $\Theta^{\mu\nu}_{\rm bound}R_\nu \neq 0$, $\Theta^{\mu\nu}_{\rm bound}$ propagates slower than light. Unlike $\Theta^{\mu\nu}_{\rm rad}$, which

detaches from the source, $\Theta^{\mu\nu}_{\text{bound}}$ remains bound to the source [7]. In other words, the source carries the four-momentum P^{μ}_{bound} along with its motion.

From (24) follows that the measure $d\sigma_\lambda \Theta^{\lambda\mu}_{\text{bound}}$ is ill-defined. However, observing that

$$p^{\mu}_0 = \int_\Sigma d\sigma_\lambda\, t^{\lambda\mu} = m_0 \dot{z}^{\mu}, \tag{25}$$

one may render m_0 a singular function of ϵ, $m_0(\epsilon)$, add Eqs. (24) and (25) up, and carry out the renormalization of mass, Eq. (11), to complete the definition of the measure $\text{Reg}_\epsilon\, d\sigma_\lambda \left(\Theta^{\lambda\mu}_{\text{bound}} + t^{\lambda\mu}\right)$ in the limit $\epsilon \to 0$, and eventually arrive at

$$p^{\mu} = \lim_{\epsilon\to 0} \text{Reg}_\epsilon \int_\Sigma d\sigma_\lambda \left(\Theta^{\lambda\mu}_{\text{bound}} + t^{\lambda\mu}\right) = m\dot{z}^{\mu} - \frac{2}{3}\, e^2 \ddot{z}^{\mu}. \tag{26}$$

This four-momentum, originally deduced in [7], is to be attributed to the *dressed* particle.

We now integrate (17) over a domain of spacetime bounded by two spacelike surfaces Σ' and Σ'', separated by a small timelike interval, with both normals directed towards the future, and a tube T_R of large radius R. Applying the Gauss–Ostrogradskiĭ theorem, we obtain[2]:

$$\left(\int_{\Sigma''} - \int_{\Sigma'} + \int_{T_R}\right) d\sigma_\mu \left(\Theta^{\lambda\mu} + t^{\lambda\mu}\right)$$

$$= \left\{\lim_{\epsilon\to 0}\left[m_0(\epsilon) + \frac{e^2}{2\epsilon}\right]\dot{z}^{\lambda} - \frac{2}{3}\, e^2 \ddot{z}^{\lambda}\right\} \Delta s - \int_s^{s+\Delta s} d\tau \left[\frac{2}{3}\, e^2 \ddot{z}^2(\tau)\dot{z}^{\lambda}(\tau) + eF^{\lambda\mu}_{\text{ext}}(z)\dot{z}_\mu(\tau)\right] = 0,$$

or, in a concise form [3],

$$\Delta p^{\lambda} + \Delta \mathcal{P}^{\lambda} + \Delta \wp^{\lambda} = 0. \tag{27}$$

Evidently (27) is identical to the Lorentz–Dirac equation (12).

On the other hand, (27) is the desired *energy–momentum balance*: the four-momentum $\Delta\wp^{\lambda} = -eF^{\lambda\mu}_{\text{ext}}\, \dot{z}_\mu\, \Delta s$ which is extracted from the external field $F^{\lambda\mu}_{\text{ext}}$ during the period of time Δs is distributed between the four-momentum of the dressed particle Δp^{λ} and the four-momentum carried away by radiation $\Delta\mathcal{P}^{\lambda}$.

Of particular interest is the case $F^{\lambda\mu}_{\text{ext}} = 0$,

$$\Delta p^{\lambda} = -\Delta \mathcal{P}^{\lambda}. \tag{28}$$

It immediately follows that the rate of change of the energy–momentum of a dressed particle, $\dot{p}^{\lambda}$, is equal to the negative of the Larmor emission rate, $-\dot{\mathcal{P}}^{\lambda}$. Here, two remarks are in order. First, $-\Delta\mathcal{P}^{\lambda}$ is a mere four-momentum[3] (rather than four-force), and hence the fact that $\dot{\mathcal{P}}^{\lambda}$ is not orthogonal to $\dot{z}^{\lambda}$ presents no special problem. Second, the energy of a dressed particle is *indefinite*,[4]

$$p^0 = m\gamma\left(1 - \tau_0 \gamma^3 \mathbf{a}\cdot\mathbf{v}\right), \tag{29}$$

where γ is the Lorentz factor $\gamma = (1 - \mathbf{v}^2)^{-1/2}$. Therefore, increasing $|\mathbf{v}|$ need not be accomplished by increasing p^0. For instance, one may readily check that the energy of a dressed particle executing

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[2] We assume that $F^{\lambda\mu}_{\text{ext}}(x)$ disappears at spatial infinity. Therefore, the only term contributing to the integral over T_R is $\Theta^{\mu\nu}_{\text{rad}}$. Taking into account the second equation of (21), the integral of $\Theta^{\mu\nu}_{\text{rad}}$ over T_R can be converted into the integral over T_ϵ, so that the upshot is given by Eq. (22).

[3] A possible interpretation of Eq. (28) is that the dressed particle experiences a jet thrust in response to emitting the electromagnetic field momentum, a kind of apparent force applied to the same point in which the emission occurs.

[4] The fact that p^0 is not positive definite is scarcely surprising. Recall that p^{μ} is the sum of two vectors $p^{\mu} = m_0\dot{z}^{\mu} + P^{\mu}_{\text{bound}}$. The bound four-momentum P^{μ}_{bound} is a timelike future-directed vector, while the four-momentum of a bare particle $m_0\dot{z}^{\mu}$ is a timelike past-directed vector because $m_0(\epsilon) < 0$ for small ϵ, as (11) suggests. Assuming that $m_0\dot{z}^{\mu} + P^{\mu}_{\text{bound}}$ is a timelike vector, one recognizes that the time component of this vector can have any sign.

a self-accelerated motion (15) steadily decreases, which exactly compensates the increase in energy of the electromagnetic field emitted.[5]

3. The Yang–Mills–Wong theory

The Yang–Mills–Wong theory describes the classical interaction of particles carrying non-Abelian charges with the corresponding Yang–Mills field [9]. A system of K such particles (thereafter called quarks) interacting with the SU($\mathcal{N}$) Yang–Mills field is governed by the action [10]

$$S = -\sum_{I=1}^{K} \int d\tau_I \left\{ m_0^I \sqrt{\dot{z}_I \cdot \dot{z}_I} + \sum_{a=1}^{\mathcal{N}^2-1} \sum_{i,j=1}^{\mathcal{N}} q_I^a \, \eta_{Ii}^* \left[\delta_j^i \frac{d}{d\tau_I} + \dot{z}_I^\mu \left(A^a{}_\mu \, T_a \right)_j^i \right] \eta_I^j \right\}$$

$$- \frac{1}{16\pi} \int d^4x \, G_a^{\mu\nu} G^a_{\mu\nu}, \tag{31}$$

where T_a are generators of SU($\mathcal{N}$), $G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + i f^a_{bs} A^b_\mu A^c_\nu$ is the field strength, f_{abc} are the structure constants of SU($\mathcal{N}$) thereafter called the colour gauge group.

Quarks, labelled with I, possess colour charges Q_I in the adjoint representation of SU($\mathcal{N}$), $Q_I = Q_I^a \, T_a$. These quantities can be expressed in terms of the basic variables η_{Ij} in the fundamental representation,

$$Q_I = \sum_{a=1}^{\mathcal{N}^2-1} \sum_{i,j=1}^{\mathcal{N}} q_I^a \, \eta_{Ii}^* (T_a)_j^i \eta_I^j. \tag{32}$$

The Euler–Lagrange equations for η and η^* read

$$\dot{\eta}^i = -(\dot{z} \cdot A^a)(T_a)^i_j \eta^j,$$

$$\dot{\eta}^*_j = \eta^*_i \, (\dot{z} \cdot A^a)(T_a)^i_j. \tag{33}$$

They can be combined into the Wong equation for the colour charge evolution [9],

$$\dot{Q}_a = -i f_{abc} \, Q^b \left(\dot{z} \cdot A^c \right). \tag{34}$$

It is convenient to rescale the colour variables: $Q \to -ig \, Q^a T_a$, $A_\mu \to (i/g) A^a_\mu T_a$, where g is the Yang–Mills coupling constant. Then Eq. (34) becomes

$$\dot{Q} = ig \left[Q, \, \dot{z}^\mu A_\mu \right]. \tag{35}$$

[5] The solution (15) is usually thought of as a pathological trait of the Lorentz–Dirac equation (12) for two main reasons: (i) this solution seems incompatible with energy conservation, and (ii) there is no experimental evidence for self-accelerated motions in the Nature. Both accusations are unjust. The fact that energy–momentum is conserved in this motion has just now been established. As to the manifestation of this phenomenon, the universe as a whole exhibiting accelerated expansion provides an excellent potential example of a free entity (brane?) which executes exponentially accelerated motion with the characteristic time equal to the inverse of current Hubble scale and emits gravitational radiation [8]. Why is the self-accelerated motion of charged particles never observed? It follows from (26) that

$$p^2 = m^2 \left(1 + \tau_0^2 \, a^2 \right). \tag{30}$$

If $\tau_0^2 \, a^2 < -1$, the dressed charged particle turns to a tachyonic state $p^2 < 0$. Let the particle be moving in the self-accelerated regime (15). Then, after a lapse of time $\Delta t = -\tau_0 \log \tau_0 |w_0|$, the critical acceleration $|a^2| = \tau_0^{-2}$ is exceeded, and the four-momentum of this dressed particle becomes spacelike. The period of time Δt over which a self-accelerated electron possesses timelike four-momenta is estimated at $\tau_0 \sim 10^{-23}$ s for electrons, and still shorter for more massive charged elementary particles. All primordial self-accelerated particles with such τ_0's have long been in the tachyonic state. However, we have not slightest notion of how tachyons can be experimentally recorded [8]. Noteworthy also is that non-Galilean and Galilean regimes of motion are never interconvertible: the history of a particular dressed charged particle is decided by the asymptotic condition in the limit $s \to \infty$. It is then conceivable that the Galilean form of evolution, corresponding to $w_0 = 0$ in Eq. (15), may well be assigned to all dressed charged particles.

It follows that the colour charge Q shares with a top the property of precessing around some axis in the colour space.

Varying z^μ and A^μ in the action (31) gives the dynamical equations for respectively the Yang–Mills field and quarks:

$$D^\lambda G_{\lambda\mu} = 4\pi g \sum_{I=1}^{K} \int_{-\infty}^{\infty} d\tau_I \, Q_I(\tau_I)\, \dot{z}^I_\mu(\tau_I)\, \delta^4\left[x - z_I(\tau_I)\right], \tag{36}$$

$$m^I_0 \ddot{z}^\lambda_I = \dot{z}^I_\mu \operatorname{tr}\left[Q_I\, G^{\lambda\mu}(z_I)\right]. \tag{37}$$

In contrast to the electric charge e, which is a constant, the colour charge Q is a dynamical variable governed by the Wong equation (35). Note, however, that the colour charge magnitude is a constant of motion,

$$\frac{d}{ds}|Q|^2 = 2\dot{Q}^a Q_a = 0, \tag{38}$$

which can be readily seen from (34) written in the Cartan basis in which $f_{abc} = -f_{bac}$. Furthermore, there is good reason to look for solutions of the Yang–Mills equation (36) satisfying the condition

$$Q^a(s) = \text{const}. \tag{39}$$

Abandoning this condition would pose the problem of an infinitely rapid precession of Q in view of the fact that the retarded field A_μ is singular on the world line.

It is clear from Eqs. (36) and (37) that the action–reaction principle holds in the Yang–Mills–Wong theory. If one conceives that only a single quark is in the universe, then Q measures both the variation of the quark state for a given field state and variation of the field state for a given quark state.

Again, we look at self-interaction for revealing the relation between the action–reaction principle and energy–momentum conservation in an explicit form. The strategy here copies that in the Maxwell–Lorentz electrodynamics, but has several traits associated with the fact that the field equation (36) is nonlinear.

There are *two kinds of retarded solutions* to the Yang–Mills equations, Abelian and non-Abelian [11]. We first turn to the simplest case that the SU(2) Yang–Mills field is generated by a single quark moving along an arbitrary timelike smooth world line [12]. The retarded Abelian solution

$$A^\mu = qT_3 \frac{\dot{z}^\mu}{\rho} \tag{40}$$

resembles the Liénard–Wiechert solution of the Maxwell–Lorentz electrodynamics, whereas the retarded non-Abelian solution is given by

$$A^\mu = \mp \frac{2i}{g} T_3 \frac{\dot{z}^\mu}{\rho} + i\kappa\,(T_1 \pm iT_2)\, R^\mu. \tag{41}$$

Here T_a, $(a = 1, 2, 3)$, are the generators of SU(2), and q and κ are arbitrary real nonzero parameters.

A remarkable feature of retarded non-Abelian solutions bearing on our discussion is that the Yang–Mills equations determine not only the field, but also the colour charge that generates this field, as exemplified by (41). This solution admits only a *single* value for the magnitude of the colour charge carried by the quark [12,13],

$$|Q|^2 = -\frac{4}{g^2}. \tag{42}$$

Recall that the electric charge e of any particle in the Maxwell–Lorentz electrodynamics may be arbitrary. The selection of a special magnitude for the colour charge of the source takes place also for a closed system of K quarks evolving in the non-Abelian regime [11],

$$\operatorname{tr}(Q_I^2) = -\frac{4}{g^2}\left(1 - \frac{1}{\mathcal{N}}\right), \quad \mathcal{N} \geq 3. \tag{43}$$

Clearly this feature of the non-Abelian dynamics offers no danger to the fulfilment of the action–reaction principle.

In the *Abelian regime*, the field equation (36) linearizes,[6] and hence, their retarded solution shows up as that in (9)–(10). All results of the previous section are reproduced with the only replacement $e^2 \to q^2$. The degrees of freedom appearing in the action (31) are rearranged on the extremals subject to the retarded condition to give a dressed quark and Yang–Mills radiation, closely resembling such entities in electrodynamics. The behaviour of a dressed quark is governed by the Lorentz–Dirac equation (12), which can be converted to the local energy–momentum balance (27).

In the *non-Abelian regime*, the field equation (36) remains nonlinear, and superposing their solutions ceases to be true. Aside from the one-quark solution (41), there is need to examine K-quark solutions, $K \geq 2$. A consistent Yang–Mills–Wong theory can be formulated for the colour gauge group SU($\mathcal{N}$) with $\mathcal{N} \geq K+1$ [11]. As an illustration we refer to a retarded SU(3) field due to two quarks [14,15],

$$A^\mu = \mp\frac{2i}{g}\left(H_1\,\frac{\dot z_1^\mu}{\rho_1} + g\,\kappa\,E_{13}^{\pm}\,R_1^\mu\right) \mp \frac{2i}{g}\left(H_2\,\frac{\dot z_2^\mu}{\rho_2} + g\,\kappa\,E_{23}^{\pm}\,R_2^\mu\right). \tag{44}$$

Here, H_a and E_{ab} are generators of SU(3) in the Cartan–Weyl basis, which are expressed in terms of the Gell-Mann matrices as follows:

$$H_1 = \frac{1}{2}\left(\lambda_3 + \frac{\lambda_8}{\sqrt{3}}\right),\ H_2 = -\frac{1}{2}\left(\lambda_3 - \frac{\lambda_8}{\sqrt{3}}\right),\ E_{13} = \frac{1}{2}\left(\lambda_4 + i\lambda_5\right),\ E_{13} = \frac{1}{2}\left(\lambda_6 + i\lambda_7\right). \tag{45}$$

$R_1^\mu = x^\mu - z_1^\mu(\tau_1)$ and $R_2^\mu = x^\mu - z_2^\mu(\tau_2)$ are, respectively, the four-vectors drawn from points $z_1^\mu(\tau_1)$ and $z_2^\mu(\tau_2)$ on the world lines of quarks 1 and 2, where the signals were emitted, to the point x^μ, where the signals were received.

Observing that A^μ is the *sum* of two single-quark terms, one may wonder of how the nonlinearity of the Yang–Mills equations is compatible with this fact. The answer is simple: two single-quark vector potentials with the *fixed* magnitudes of the colour charges, as shown in Eq. (43), are combined in Eq. (44), but it is impossible to build solution as an arbitrary superposition of these terms. If either of them is multiplied by a coefficient different from 1 and added to another, no further solution arises.

Due to this feature – which is characteristic of the general K-quark case – we have

$$\Theta^{\mu\nu} = \sum_I \left(\Theta_I^{\mu\nu} + \sum_{J\neq I}\Theta_{IJ}^{\mu\nu}\right), \tag{46}$$

where $\Theta_I^{\mu\nu}$ is comprised of the field generated by the Ith quark, and $\Theta_{IJ}^{\mu\nu}$ contains mixed contributions of the fields due to the Ith and Jth quarks. Furthermore, $\Theta_I^{\mu\nu}$ splits into bound and radiated parts. Every term of Eq. (46) satisfies the local conservation law of the type shown in Eq. (21), and hence represents a dynamically independent part of $\Theta^{\mu\nu}$. The stress–energy tensor $\Theta^{\mu\nu}$ is thus *similar in structure* to that in the Maxwell–Lorentz theory.

We now restrict our attention to a *single* quark of this K-quark system. For notational convenience, we omit the quark labelling.

Using the line of reasoning developed in the previous section, and observing that the linearly rising term of A_μ does not contribute to $\Theta^{\mu\nu}$ because

$$\mathrm{tr}\,(H_l\,E_{mn}^{\pm}) = 0, \quad \mathrm{tr}\left(E_{kl}^{\pm}\,E_{mn}^{\pm}\right) = 0, \tag{47}$$

we arrive at the conclusion that the four-momentum of the retarded Yang–Mills field generated by the quark under study is given by $P^\mu = P^\mu_{\text{bound}} + \mathcal{P}^\mu$, where P^μ_{bound} and $\mathcal{P}^\mu$ are respectively the bound and radiated parts of this four-momentum.

[6] The reason for this linearization is that the colour field variables are restricted to the Cartan subalgebra of the Lie algebra of the associated gauge group. Since the Lie algebra su ($\mathcal{N}$) is of rank $\mathcal{N}-1$, there exist $\mathcal{N}-1$ diagonal matrices H_a forming a Cartan subalgebra of commuting matrices.

An accelerated quark emits

$$\mathcal{P}^{\mu} = -\frac{2}{3}\,\mathrm{tr}\,(Q^2)\int_{-\infty}^{s} d\tau\,\ddot{z}^2\,\dot{z}^{\mu}. \tag{48}$$

Owing to the negative norm of the colour charges, Eqs. (42) and (43), the emitted energy is negative, which suggests that the quark gains, rather than loses, energy by emitting the Yang–Mills radiation in the non-Abelian regime. An explicit calculation shows that this is indeed the case:

$$\dot{\mathcal{P}}\cdot\dot{z} = \frac{8}{3g^2}\left(1-\frac{1}{\mathcal{N}}\right)\ddot{z}^2 < 0. \tag{49}$$

This phenomenon might be interpreted as absorbing convergent waves of positive energy rather than emitting divergent waves of negative energy [13].

Adding the bound part of the field four-momentum

$$P^{\mu}_{\mathrm{bound}} = \mathrm{tr}\,(Q^2)\left(\frac{1}{2\epsilon}\,\dot{z}^{\mu} - \frac{2}{3}\,\ddot{z}^{\mu}\right) \tag{50}$$

to the mechanical four-momentum $p^{\mu}_0 = m_0\dot{z}^{\mu}$, and carrying out the renormalization of mass in a way similar to (11), gives the four-momentum of a dressed quark

$$p^{\mu} = m\,(\dot{z}^{\mu} + \ell\,\ddot{z}^{\mu})\,. \tag{51}$$

Here, m is the renormalized mass, and

$$\ell = \frac{8}{3mg^2}\left(1-\frac{1}{\mathcal{N}}\right) \tag{52}$$

is a characteristic length inherent in the non-Abelian dynamics of this dressed quark.

The mixed terms in Eq. (46) have integrable singularities ρ^{-2} on every world line. Their treatment is therefore similar to that of $\Theta^{\mu\nu}_{\mathrm{mix}}$ in the Maxwell–Lorentz electrodynamics. The integration of these terms gives

$$\wp^{\mu} = -\int_{-\infty}^{s} d\tau\, f^{\mu}_{\mathrm{ext}}[z(\tau)]\,, \tag{53}$$

where the integrand is the colour four-force exerted on the given quark by all other quarks at the instant τ. The explicit form of f^{μ}_{ext} is of no concern in the present context.

We reiterate mutatis mutandis the argument of the previous section to find

$$\dot{p}^{\mu} + \dot{\mathcal{P}}^{\mu} = f^{\mu}_{\mathrm{ext}}. \tag{54}$$

According to this balance equation, the four-momentum $d\wp^{\mu} = -f^{\mu}_{\mathrm{ext}}ds$ extracted from an external field is used for changing the four-momentum of the dressed quark dp^{μ} and emitting the Yang–Mills radiation four-momentum $d\mathcal{P}^{\mu}$. A special feature of Eq. (54) is that $d\mathcal{P}^0$ is associated with emitting negative-energy waves or, what is the same – absorbing positive-energy waves.

Substitution of (51), (48) and (53) into (54) gives the equation of motion for a dressed quark

$$m\left[\ddot{z}^{\mu} + \ell\left(\dddot{z}^{\mu} + \dot{z}^{\mu}\ddot{z}^2\right)\right] = f^{\mu}_{\mathrm{ext}}, \tag{55}$$

differing from the Lorentz–Dirac equation (12) only in the overall sign of the parenthesized term and changing τ_0 by ℓ. If $f^{\mu}_{\mathrm{ext}} = 0$, the general solution of Eq. (55) is

$$\dot{z}^{\mu}(s) = V^{\mu}\cosh(\alpha_0 + w_0\ell\, e^{-s/\ell}) + U^{\mu}\sinh(\alpha_0 + w_0\ell\, e^{-s/\ell}), \tag{56}$$

where V^{μ} and U^{μ} are constant four-vectors such that $V\cdot U = 0$, $V^2 = -U^2 = 1$, and α_0 and w_0 are arbitrary parameters. A free quark may therefore execute a non-uniform motion with exponentially decreasing acceleration. The world line of this self-decelerated motion asymptotically approaches a

straight line. This situation can be interpreted in the spirit of the action–reaction principle. Eq. (54) becomes

$$dp^{\mu} = -d\mathcal{P}^{\mu}. \tag{57}$$

The free dressed quark feels the 'reverse' four-momentum transfer (responsible for the self-deceleration) because the phenomenon of radiating the Yang–Mills four-momentum is actually changed by that of absorbing this four-momentum. Nevertheless, Eq. (57) offers direct evidence that the action–reaction principle is equivalent to energy–momentum conservation on the world line.

4. Gravitation

The action–reaction principle does not hold in the gravitational interaction described by general relativity. Indeed, the coupling between a particle of mass m and the gravitational field is equal to m, so that the particle is governed by the geodesic equation

$$\frac{d^2z^{\lambda}}{d\tau^2} + \Gamma^{\lambda}_{\mu\nu}\frac{dz^{\mu}}{d\tau}\frac{dz^{\nu}}{d\tau} = 0, \tag{58}$$

which is mass-independent. On the other hand, the field equation

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} - 8\pi G_{\mathrm{N}}t^{\mu\nu} = 0 \tag{59}$$

with the delta-function source

$$t^{\mu\nu}(x) = m\int_{-\infty}^{\infty} d\tau\, \dot{z}^{\mu}(\tau)\dot{z}^{\nu}(\tau)\delta^4[x - z(\tau)] \tag{60}$$

shows that the greater the m, the stronger is the generated gravitational field. The influence of particles on the state of the gravitational field is different for different m, even though the gravitational field exerts on every particle in a uniform way, no matter what m is. This is contrary to the action–reaction principle.

Does this violation of the action–reaction principle imply that the energy–momentum conservation law is missing from general relativity? While on the subject of arbitrary curved manifolds, the idea of translational invariance is irrelevant, whence it follows that not only energy and momentum are not conserved, but also the very construction of energy and momentum suggested by Noether's first theorem is no longer defined.

To avoid this conclusion, one normally turns to field-theoretic treatments of gravity. This is feasible if the gravitational field can be granted to be 'sufficiently weak',

$$g_{\mu\nu} = \eta_{\mu\nu} + \phi_{\mu\nu}, \tag{61}$$

where $|\phi_{\mu\nu}| \ll 1$. The quantity $\phi_{\mu\nu}$ is thought of as a second-rank tensor field[7] defined in a flat background $\mathbb{R}_{1,3}$ whose symmetry properties enable us to endow the resulting dynamics with conserved energy–momentum through the standard Noether's prescription.

It is believed that general relativity leaves room for both weak and strong gravity. Strange though it may seem, a simple and convincing criterion for discriminating between *weak* and *strong* gravity still remains to be established. We therefore have to address this issue. But our concern here is not with elaborating this criterion in every respect. Rather, we only state the central idea and explicate it by the example of the Schwarzschild metric.

Intuition suggests that the strong gravity should be associated with a great warping of spacetime. However, a characteristic curvature whereby the changes in spacetime configurations might be rated as 'drastic' is absent from general relativity, sending us in search of another measure of such changes. It seems reasonable to assume that switching between weak and strong gravitational regimes is due to spacetime topology alterations. The 'strong gravitational field' is then recognized as a qualitative

[7] Recent developments in bimetric theories of gravitation is reviewed in [16].

rather than quantitative concept. The field equation (59), being differential equations, is local in character. They tell nothing about the topology of their solutions. A global solution can be recovered when its infinitesimal pseudoeuclidean fragments are assembled into an integral dynamical picture, and the topology of this solution may well differ from the topology of Minkowski spacetime if the assembly is subject to a restrictive boundary condition. To illustrate, we refer to the Schwarzschild metric [17],

$$ds^2 = -\left(1-\frac{r_S}{r}\right)dt^2 + \left(1-\frac{r_S}{r}\right)^{-1}dr^2 + r^2 d\Omega, \tag{62}$$

where, $d\Omega$ is the round metric in a sphere S^2, and $r_S = 2G_N M$ is the Schwarzschild radius. A 3-dimensional spacelike surface Σ_3 endowed with this metric has a twofold geometric interpretation. First, it looks like a 'bridge' between two otherwise Euclidean spaces, and, second, it may be regarded as the 'throat of a wormhole' connecting two distant regions in one Euclidean space in the limit when this separation of the wormhole mouths is very large compared to the circumference of the throat [18].

To describe a curved manifold $\mathcal{M}$, a set of overlapping coordinate patches covering $\mathcal{M}$ is called for. If one yet attempts to use a single coordinate patch, a singularity in the resulting description can arise. The gravitation is amenable to a field-theoretic treatment until the mapping of the metric $g_{\mu\nu}$ into the field $\phi_{\mu\nu}$, as shown in (61), is bijective and smooth, which is the same as saying that every curved spacetime configuration, associated with some gravitational effect, can be smoothly covered with a single coordinate patch. In contrast, for a manifold whose topology is nontrivial, the quest for a single-patch covering culminates in a singular boundary, bearing some resemblance to a shock wave, as exemplified by the Schwarzschild metric (62) in which the coefficient of dr^2 becomes singular at $r = r_S$, so that this solution exhibits a standing spherical shock wave of the gravitational field.

One may argue that this is an apparent singularity, related to the choice of coordinates, because the curvature invariants are finite and well behaved at $r = r_S$, and, furthermore, the equation for the geodesics (58) shows a singular behaviour only at $r = 0$. There are coordinates u and v, proposed in [19] and [20],

$$u = \sqrt{\frac{r}{r_S}-1}\exp\left(\frac{r}{r_S}\right)\cosh\left(\frac{t}{r_S}\right), \quad v = \sqrt{\frac{r}{r_S}-1}\exp\left(\frac{r}{r_S}\right)\sinh\left(\frac{t}{r_S}\right), \tag{63}$$

such that the Schwarzschild metric, being written in terms of u and v, is regular in the whole (u, v) plane, except for the point $v^2 - u^2 = 1$ corresponding to $r = 0$.

In response to this objection, we would note that the introduction of these u and v is a clever trick to drive the shock wave in the singular point $r = 0$. However, our prime interest is with the very existence of a shock wave, as evidence of the nontrivial topology, rather than its position in a particular coordinate system. The apparent regularity of the metric everywhere except $r = 0$ is due to an unfortunate choice of coordinates which hides the Schwarzschild shock wave.

The 'floating' position of the shock wave makes it clear that the strong gravitational regime is unrelated to the magnitude of field variables. It is a topologically nontrivial affair which renders the regime strong.

This brings up the question as to whether the violation of regular behaviour of the Schwarzschild metric at $r = r_S$ is an artefact of the original Schwarzschild description. Some fifty years ago people were inclined to believe that such is indeed the case. By now, however, r_S is recognized as an objectively existing entity to characterize the event horizon of an isolated spherically symmetric stationary black hole. The event horizon of a Schwarzschild black hole shows a demarcation between spacetime regions characterized by opposite signatures of the metric.[8] This geometrical layout, if it exists, provides an explicit scheme for interfacing the classical and the quantum [21].

Let us take a closer look at why energy and momentum are to be regarded as poorly defined concepts. General relativity allows the Hamiltonian formulation for at least such systems whose

[8] Note that the only quantity which has a discontinuity jump at the front of a strong gravitational shock wave is the signature because the metric immediately anterior and posterior to the front can be brought to either of two diagonal forms: diag$(+---)$ or diag$(-+++)$.

geometric rendition is compatible with the idea of asymptotically flat spacetime [22–24]. More specifically, one supposes that the metric $g_{\mu\nu}$ approaches the Lorentz metric $\eta_{\mu\nu}$ at spatial infinity sufficiently rapidly, namely

$$g_{\mu\nu} = \eta_{\mu\nu} + O\left(\frac{1}{r}\right), \quad \partial_\lambda g_{\mu\nu} = O\left(\frac{1}{r^2}\right), \quad r \to \infty. \tag{64}$$

The second condition is claimed to be needed so that the Lagrangian

$$L = \int d^3\mathbf{x}\, \mathcal{L}(t, \mathbf{x}) \tag{65}$$

with the commonly used first order Lagrangian density of the gravitational field sector

$$\mathcal{L} = \sqrt{-g}\, g^{\mu\nu} \left(\Gamma^\sigma_{\mu\nu}\, \Gamma^\lambda_{\sigma\lambda} - \Gamma^\lambda_{\mu\sigma}\, \Gamma^\sigma_{\nu\lambda}\right) \tag{66}$$

should converge. The volume integral in (65) diverges for the Schwarzschild solution expressed in terms of the original Schwarzschild coordinates, appearing in (62), because $\mathcal{L} = O(1)$ as $r \to \infty$. In contrast, the use of isotropic coordinates, which recasts the Schwarzschild metric (62) into

$$ds^2 = -\frac{1 - \frac{r_S}{4r}}{1 + \frac{r_S}{4r}}\, dt^2 + \left(1 + \frac{r_S}{4r}\right)\left(dx^2 + dy^2 + dz^2\right), \tag{67}$$

results in $\mathcal{L} = O(1/r^4)$, and hence affords the convergence of the Lagrangian (65). It follows from this simple example that both the asymptotical flatness

$$R^\alpha_{\beta\gamma\delta} \to 0, \quad r \to \infty, \tag{68}$$

and a good choice of coordinates shares in the responsibility for the convergence of additive quantities such as the Lagrangian and Hamiltonian.

With the neat Hamiltonian formulation developed in [22–24], one would expect that the total energy–momentum P^μ has an unambiguous significance. We now examine the correctness of this expectation restricting ourselves to P^0 for simplicity.

The total energy is given by the numerical value of the Hamiltonian

$$E = \int d^3\mathbf{x}\, \mathcal{H}\,(t, \mathbf{x})\,. \tag{69}$$

$\mathcal{H}$ is a cumbersome construction which is immaterial for our discussion. However, the key part of this construction proves to be cast [25] in a convenient form,

$$E = \frac{1}{16\pi} \oint dS_j \left(\frac{\partial}{\partial x_i}\, g_{ij} - \frac{\partial}{\partial x_j}\, g_{ii}\right). \tag{70}$$

Here, the integral is evaluated over a 2-dimensional surface at spatial infinity.

It is possible to prove [26,27] that an isolated gravitating system having non-negative local mass density has non-negative total energy E. For example, for the Schwarzschild configuration generated by a point particle of mass m the surface integral (70) is easily evaluated to give

$$E = m. \tag{71}$$

Could the condition (64) be relaxed so that the asymptotical flatness condition (68) would hold, and every pertinent additive quantity in this Hamiltonian formulation remains convergent? To be more precise, we proceed from the metric $g_{\mu\nu}$ exhibiting the asymptotic behaviour (64), and transform the initial spatial coordinates x^i into new ones $\bar{x}^i$,

$$x^i = \bar{x}^i\,[1 + f(\bar{r})]\,, \tag{72}$$

where f is an arbitrary regular function subject to the following conditions:

$$f(\bar{r}) \geq 0, \quad \lim_{\bar{r}\to\infty} f(\bar{r}) = 0, \quad \lim_{\bar{r}\to\infty} \bar{r} f'(\bar{r}) = 0. \tag{73}$$

For the mapping (72) to be bijective, the condition

$$\frac{\partial r}{\partial \bar{r}} = 1 + f(\bar{r}) + \bar{r} f'(\bar{r}) > 0 \tag{74}$$

is necessary and sufficient. Indeed, with (74), the mapping is explicitly invertible,

$$J = \det\left(\frac{\partial x}{\partial \bar{x}}\right) = [1 + f(\bar{r})]\,\frac{\partial r}{\partial \bar{r}} \neq 0. \tag{75}$$

One such example [28] is

$$f(\bar{r}) = 2\,\alpha^2 \sqrt{\frac{l}{\bar{r}}}\left[1 - \exp\left(-\frac{\epsilon^2\,\bar{r}}{l}\right)\right], \tag{76}$$

where α and ϵ are arbitrary nonzero numbers, and l is an arbitrary parameter of dimension of length. This is a bijective monotonically increasing regular mapping $r \to \bar{r}$ which becomes 1 as $\epsilon \to 0$. The leading asymptotical terms of spatial components of the metric and those of the Christoffel symbols can be shown [28] to be

$$g_{ij} = \delta_{ij} + O\left(\frac{1}{\bar{r}^{1/2}}\right), \quad \Gamma^i_{jk} = O\left(\frac{1}{\bar{r}^{3/2}}\right), \quad \bar{r} \to \infty, \tag{77}$$

while the Lagrangian density behaves as

$$\mathcal{L} = O\left(\frac{1}{\bar{r}^{7/2}}\right), \quad \bar{r} \to \infty. \tag{78}$$

This provides the convergence of the volume integral in (65). Other additive quantities prove to be convergent as well.[9]

The mapping (72) with f defined in (76) is instructive to apply to the Schwarzschild metric which is initially written in terms of isotropic coordinates (67). One can show [28] that the total energy of the Schwarzschild configuration generated by a point particle of mass m takes any positive values, greater than, or equal to m, when α^2 runs through $\mathbb{R}_+$,

$$E = m\left(1 + \alpha^4\right). \tag{79}$$

We thus see that the total energy of gravitational systems with nontrivial topological contents depends on the foliation of spacetime. The Schwarzschild solution expressed in terms of coordinates for which the asymptotic condition is given in a relaxed form, Eq. (77), is a good case in point. This solution rearranges the initial degrees of freedom appearing in the Lagrangian density (66) to yield a coordinatization-dependent expression for the total energy functional (70). The same is true of the associated momentum.

The situation closely parallels that in the paradoxial Banach–Tarski theorem which states [29]: given a unit ball in three dimensions, there exists a decomposition of this ball into a finite disjoint subsets which can then be reassembled through continuous movements of the pieces, without running into one another and without changing their shape, to yield another ball of larger radius. These situations share a common trait in that both the Banach–Tarski decomposition and the Schwarzschild black hole formation are due to topological rearrangements which are responsible for making the three-dimensional measures of the resulting geometrical layouts poorly defined. The measure appearing in the Banach–Tarski theorem is the ordinary volume of the balls (more precisely, Lebesgue measure), while the measure in the gravitational energy problem is that of the functional (69). When turning to the surface integral for calculation of the total energy, Eq. (70), there arises the situation which may be likened to that of the Hausdorff paradox on enlarging spheres [30].

The usual inference that the Banach–Tarski partitioning procedure has nothing to do with physical reality because there is no material ball which is not made of atoms overlooks one important instance—black holes. Each isolated, stationary black hole is completely specified by three parameters:

[9] Note also that the asymptotical flatness, Eq. (68), is still the case.

its mass m, angular momentum J, and electric charge e. Whatever the content of a system which collapses under its own gravitational field, the exterior of the resulting black hole is described by a Kerr–Newman solution. All initial geometric features of the collapsing system, except for those peculiar to a perfect ball, which may possibly rotate and carry electric charge, disappear in the black hole state [31,32]. Furthermore, the event horizon which is meant for personifying the black hole is devoid of the grain structure that was inherent in the collapsing system.

5. Discussion and outlook

In Sections 2 and 3 we saw that a careful analysis of the self-interaction problem may give an insight into the relation between the action–reaction principle and energy–momentum conservation provided that the rearrangement of degrees of freedom is taken into account. Umezawa [33] was the first to put the term 'rearrangement' in circulation by the example of spontaneous symmetry breaking. The mechanism for rearranging classical gauge fields was further studied in [13,11,3]. While a precise formulation of this mechanism is still an open problem, the intuitive idea underlying the rearrangement is quite simple. In choosing variables for the description of a field system, preference is normally given to those which are best suited for realizing all supposed fundamental symmetries of the action. But some degrees of freedom so introduced may be unstable. This gives rise to reassembling the initial degrees of freedom into new, stable aggregates whose dynamics is invariant under broken or deformed groups of symmetries. Aggregates obeying the usual requirement of stability

$$\delta S = 0, \quad \delta^2 S > 0 \tag{80}$$

form readily in field theories affected by spontaneous symmetry breaking, as exemplified by the Goldstone and Higgs models. However, this criterion for discriminating between stable and unstable modes is difficult if not impossible to apply to local field theories with delta-function sources owing to divergences arising in the self-interaction problem. We thus have to look for alternative criteria.

Let us return to the Maxwell–Lorentz electrodynamics. We take, as the starting point, the on-shell dynamics of a bare particle and electromagnetic field engendered by the equations of motion $\varepsilon^\lambda = 0$ and $\mathcal{E}_\mu = 0$, Eqs. (4) and (3), together with the retarded boundary condition. However, this dynamics blows up on the world line, which, in view of Eq. (5), is tantamount to stating that the measure $T^{\mu\nu}d\sigma_\nu$ is ill defined. It would be tempting to construe such divergences as evidences of instability.

We then divide $T^{\mu\nu}d\sigma_\nu$ into a well-defined part and the remainder. But this separation is ambiguous: an arbitrary regular term can be added to one part and subtracted from the other to give an equivalent separation. To fix the separation, we impose the condition that every term obeys the local conservation law (21). The functionals (22) and (23), expressing, respectively, the four-momentum radiated by the charge and four-momentum extracted from a free field, refer to the well-defined part of $T^{\mu\nu}d\sigma_\nu$. We complete the definition of $\left(\Theta^{\mu\nu}_{\text{bound}} + t^{\mu\nu}\right) d\sigma_\nu$ by carrying out the renormalization of mass, Eq. (11). The functional (26) is regarded as the four-momentum of a dressed charged particle. As might be expected, the rearrangement outcome, the Lorentz–Dirac equation $\Lambda^\mu = 0$, Eq. (12), governing the behaviour of the dressed particle, is depleted of some symmetries embedded in the action. Indeed, this dynamical equation is not invariant under time reversal $s \to -s$.

We thus come to a new on-shell dynamics in which the equation of motion for a bare particle $\varepsilon^\mu = 0$ is replaced with the equation of motion for a dressed particle $\Lambda^\mu = 0$, and all relevant integral quantities are well defined. Therefore, the rearrangement of the Maxwell–Lorentz electrodynamics can be briefly outlined as follows: since the on-shell dynamics which arises from extremizing the action and imposing the retarded boundary condition is divergent, the initial degrees of freedom appearing in the action are induced to reassemble into new aggregates governed by a tractable dynamics. What are the ways open to this reassembly?

The arena for rearranging the Maxwell–Lorentz electrodynamics is a line $\mathbb{R}$ covered by the evolution variable τ which parametrizes the world line, and a plane $\mathbb{E}_2$ spanned by two vectors R^μ and V^μ used in defining the retarded Liénard–Wiechert 2-form F, Eq. (9). Reparametrization invariance of the action and local $\mathrm{SL}(2,\mathbb{R})$ invariance of the 2-form F control the rearrangement scenario. Hence, the ways open to the reassembly are specified by the properties of the local translation group T responsible for reparametrizations and the local $\mathrm{SL}(2,\mathbb{R})$ group acting in the retarded field plane.

Recall the main implication of reparametrization invariance, Noether's second theorem [2]. It is convenient to restrict our consideration to an infinitesimal reparametrization

$$\delta\tau = \epsilon(\tau), \tag{81}$$

where ϵ is an arbitrary smooth function of τ close to zero, which becomes vanishing at the end points of integration. Variation of τ implies the corresponding variation of the world line coordinates

$$\delta z^\mu = \dot{z}^\mu \epsilon. \tag{82}$$

In response to the reparametrization (81)–(82), the action varies as

$$\delta S = \int d\tau\, \varepsilon_\mu \dot{z}^\mu \epsilon. \tag{83}$$

Let S be invariant under reparametrizations, $\delta S = 0$. Because ϵ is assumed to be an arbitrary function τ, one concludes that

$$\dot{z}^\mu \varepsilon_\mu = 0. \tag{84}$$

This equation is a manifestation of Noether's second theorem: invariance of the action under the transformation group (81) involving an arbitrary infinitesimal function ϵ implies a linear relation between Eulerians.

The identity (84) suggests that ε_μ contains the projection operator on a hyperplane with normal $\dot{z}^\mu$,

$$\overset{\dot{z}}{\perp}_{\mu\nu} = \eta_{\mu\nu} - \frac{\dot{z}_\mu \dot{z}_\nu}{\dot{z}^2}, \tag{85}$$

annihilating identically any vector parallel to $\dot{z}^\mu$. Reparametrization invariance bears on the projection structure of the basic dynamical law for a bare particle which can be written[10] as

$$\overset{\dot{z}}{\perp}(\dot{p} - f) = 0, \tag{86}$$

where p is the four-momentum of a bare particle, and f an external four-force.

In view of the identities $\dot{z}^2 = 1$, $\dot{z}\cdot\ddot{z} = 0$, $\dot{z}\cdot\dddot{z} = -\ddot{z}^2$, the Lorentz–Dirac equation (12) can be brought to the form of Eq. (86) in which p is the four-momentum of a dressed particle, defined in (26), and f is again an external four-force.

The structure of (86) makes it clear that a dressed particle experiences only an external force. This equation contains no term through which the dressed particle interacts with itself. The rearrangement eliminates self-interaction. The rearranged dynamical picture contains only autonomous, foreign to each other entities.

It may be worth pointing out that both equation of motion for a bare particle $\varepsilon^\mu = 0$ and equation of motion for a dressed particle $\Lambda^\mu = 0$ are generally not invariant under reparametrizations. Instead, this local symmetry leaves its imprint on the form of ε^μ and Λ^μ through the presence of the projector $\overset{\dot{z}}{\perp}$.

Invariance under the SL(2, $\mathbb{R}$) group stems from the fact that the 2-form F describing the retarded field of a single charge is proportional to $R\wedge V$, that is, F is decomposable [13,3]. Given a decomposable 2-form F, the invariant $\mathcal{P} = \frac{1}{2}F_{\mu\nu}{}^*F^{\mu\nu}$ is identically zero. As for the invariant $\mathcal{S} = \frac{1}{2}F_{\mu\nu}F^{\mu\nu}$, using (9)–(10), we find $\mathcal{S} = -e^2/\rho^4$. Therefore, a single charge moving along an arbitrary timelike world line generates the retarded field $F^{\mu\nu}$ of electric type. In other words, whatever the motion of the charge, there is a Lorentz frame of reference, special for each point x^μ, such that only electric field persists, more precisely, $|\mathbf{E}| = e/\rho^2$ and $\mathbf{B} = 0$.

Rewrite (9) as

$$F = \frac{e}{\rho^2}\varpi, \quad \varpi = R\wedge V. \tag{87}$$

[10] The projector $\overset{\dot{z}}{\perp}$ may arise in (86) from a completely different origin, namely smooth embedding of Newtonian dynamics into sections of Minkowski space perpendicular to the world line [3].

A pictorial rendition of the bivector ϖ is the parallelogram of the vectors R^μ and V^μ. The area A of the parallelogram is

$$A = \sqrt{-V^2\left(\overset{V}{\perp} R\right)^2} = V \cdot R = 1. \tag{88}$$

The bivector ϖ is invariant under the special linear group of real unimodular 2×2 matrices $\mathrm{SL}(2, \mathbb{R})$ which rotate and deform the initial parallelogram, converting it to parallelograms of unit area belonging to the plane spanned by the vectors R^μ and V^μ. Therefore, ϖ is independent of concrete directions and magnitudes of the constituent vectors R^μ and V^μ. ϖ depends only on the parallelogram's orientation. The parallelogram can always be built from a timelike unit vector e_0^μ and a spacelike imaginary-unit vector e_1^μ perpendicular to e_0^μ, $\varpi = e_0 \wedge e_1$. In fact, there are three different cases:

(i) $V^2 > 0$,

$$e_0^\mu = \frac{V^\mu}{\sqrt{V^2}}, \quad e_1^\mu = \sqrt{V^2}\left(-R^\mu + \frac{V^\mu}{V^2}\right), \tag{89}$$

(ii) $V^2 < 0$,

$$e_0^\mu = \sqrt{-V^2}\left(R^\mu - \frac{V^\mu}{V^2}\right), \quad e_1^\mu = \frac{V^\mu}{\sqrt{-V^2}}, \tag{90}$$

(iii) $V^2 = 0$,

$$e_0^\mu = \frac{1}{\sqrt{2}}\left(\rho V^\mu + \frac{R^\mu}{\rho}\right), \quad e_1^\mu = \frac{1}{\sqrt{2}}\left(\rho V^\mu - \frac{R^\mu}{\rho}\right). \tag{91}$$

In the Lorentz frame with the time axis parallel to the vector e_0^μ, all components of $F^{\mu\nu}$ are vanishing, except for F^{01}. The formulas (89)–(91) specify explicitly a frame in which the retarded electromagnetic field generated by a single arbitrarily moving charge appears as a pure Coulomb field at each observation point. With a curved world line, this frame is noninertial.

The decomposable 2-form F is invariant under the $\mathrm{SL}(2, \mathbb{R})$ transformations which can be carried out independently at any spacetime point. Therefore, we are dealing with local invariance. This invariance is not pertinent to electrodynamics as a whole, and hence gives rise to no Noether identities. Rather, this is a property of the retarded solution $F^{\mu\nu}_{\mathrm{ret}}$, shown in Eqs. (9)–(10).[11]

Therefore, the retarded solution $F^{\mu\nu}_{\mathrm{ret}}$ is determined not only by the field as such but also by the frame of reference in which this quantity is measured. On the other hand, $\Theta^{\mu\nu}$ is not invariant under such $\mathrm{SL}(2, \mathbb{R})$ transformations; $\Theta^{\mu\nu}$ carries information about both the field and the Lorentz frame which is used to describe $F^{\mu\nu}_{\mathrm{ret}}$. Nevertheless, the functionals (22), (23) and (26) are well defined and frame-independent.

The rearrangement in the Yang–Mills–Wong theory shows a general resemblance of that in the Maxwell–Lorentz electrodynamics. The field strength generated by a single quark is also given by a decomposable 2-form F in both Abelian and non-Abelian regimes. The retarded Yang–Mills field F is always invariant under the local group $\mathrm{SL}(2, \mathbb{R})$.

A special feature of the Yang–Mills–Wong theory (as opposed to the Maxwell–Lorentz electrodynamics) is that non-Abelian regimes of evolution exhibit *spontaneously deformed* gauge symmetries [11,3]. Without going into detail, we explicate this phenomenon by the simplest example. Consider the solution (41) which describes the retarded non-Abelian field generated by a single quark in the SU(2) Yang–Mills–Wong theory. By introducing an alternative matrix basis

$$\mathcal{T}_1 = T_1, \quad \mathcal{T}_2 = iT_2, \quad \mathcal{T}_3 = T_3, \tag{92}$$

we convert this solution to the form $A_\mu = \mathcal{A}^a_\mu \mathcal{T}_a$ where all coefficients $\mathcal{A}^a_\mu$ are imaginary. Elements of this basis obey the commutation relations of the $\mathrm{sl}(2, \mathbb{R})$ Lie algebra. We thus see that the gauge

[11] The advanced field $F^{\mu\nu}_{\mathrm{adv}}$ can also be represented in a form similar to (9)–(10), that is, the 2-form F_{adv} is decomposable whereas combinations $F_{\mathrm{ret}} + \alpha F_{\mathrm{adv}}$ are not.

group of the solution (41) is actually SL(2, $\mathbb{R}$).[12] Where does this group of symmetry come from? Its origin bears no relation to spontaneous symmetry breakdown: SU(2) and SL(2, $\mathbb{R}$) are the compact and noncompact real forms of the complex group SL(2, $\mathbb{C}$). Invariance of the action under SU(2) automatically entails its invariance under the complexification of this group, SL(2, $\mathbb{C}$). The emergence of a solution invariant under a real form of SL(2, $\mathbb{C}$) different from the initial SU(2) is a rearrangement phenomenon specific to the Yang–Mills–Wong theory, called spontaneous symmetry deformation. The solutions (41) and (40) are different not only in their symmetry aspect, but also in physical manifestations, say, the former manifests itself as the Yang–Mills field of 'magnetic' type while the latter is the Yang–Mills field of 'electric' type. Dressed quark, associated with these solutions, are governed by respectively equations of motion (55) and (12), both being in agreement with the action–reaction principle.

The rearrangement of general relativity is vastly different from that of the Maxwell–Lorentz electrodynamics and Yang–Mills–Wong theory. Indeed, the total stress–energy tensor $T^{\mu\nu}$ is identical to the left-hand side of Eq. (59), that is, the on-shell $T^{\mu\nu}$ is zero. It is therefore impossible to define a three-dimensional measure weighted with $T^{\mu\nu}$. And yet, the on-shell dynamics exhibits a kind of blow-up: gravitational singularities. This troublesome feature of the theory is found even if delta-function sources are substituted by continuously distributed matter obeying a reasonable energy condition, the local positive energy condition [34–36]. However, the responsibility for the rearrangement does not rest with the divergent dynamics. Gravitational degrees of freedom are induced to reassemble into new topologically nontrivial aggregates due to instabilities which owe their origin to the failure of the action–reaction principle. The ways open to the rearrangement of general relativity are specified by the properties of four-dimensional reparametrizations

$$x^{\mu} = F^{\mu}(x'), \quad g_{\mu\nu}(x) = \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} g'_{\alpha\beta}(x'), \tag{93}$$

where F^{μ} are arbitrary smooth functions. These transformations form an infinite group, the group of *diffeomorphism* invariance implying invariance of the metric under the local Lorentz group and parallel transport group.

It would be interesting to enquire into why the functionals (69) and (70) become coordinatization-dependent for systems having nontrivial topological contents in the light of the analyses which are lumped together as the 'Banach–Tarski theorem' [37]. Note that the very analogy between the Banach–Tarski decomposition and the rearrangement of gravitational degrees of freedom may seem in doubt because the former has to do with sets of *points* in Euclidean space $\mathbb{E}_3$, whereas the latter refers to the pseudo-Riemannian *metric* structure. But the resemblance of these procedures is ensured by the fact that the study of the affair with $\mathbb{E}_3$ is actually transferred to exploring the properties of bijective mappings of sets in $\mathbb{E}_3$, and the like is true for the rearrangement of gravitational degrees of freedom. A central idea of the Banach–Tarski analyses is that if a bounded set can be decomposed in a paradoxial way with respect to a group G, then G contains *free* subgroups, in particular a ball in $\mathbb{E}_3$ is SO(3)-paradoxial because the action of SO(3) is that of a free non-Abelian isometry group [37]. The development of this idea in relation to the action of the isometry group composed of the local SO(1, 3) group and parallel transport group, having free non-Abelian subgroups, may give a plausible explanation for the fact that the measure of integral quantities such as (69) and (70) is to be poorly defined.

On the other hand, the Banach's theorem stating that no paradoxial decompositions exist in $\mathbb{R}$ and $\mathbb{E}_2$ [37] should be likened to the affair with the well-defined measures in the rearranged Maxwell–Lorentz electrodynamics and Yang–Mills–Wong theory. The class of groups whose actions preserve finitely additive, isometry-invariant measures of the bounded sets on $\mathbb{R}$ and $\mathbb{E}_2$ are known to be *amenable* groups, specifically *solvable* groups, which include Abelian groups. It is conceivable that the groups of reparametrizations and local SL(2, $\mathbb{R}$) transformations controlling the rearrangement of these theories have what amounts to the desired properties of amenable groups.

[12] This SL(2, $\mathbb{R}$) gauge group should not be confused with the SL(2, $\mathbb{R}$) symmetry transformations which leave a decomposable 2-form F unchanged. Given the initial SU($\mathcal{N}$) gauge symmetry with $\mathcal{N} \geq 2$, the spontaneously deformed gauge symmetry is found to be embedded in the SL($\mathcal{N}$, $\mathbb{R}$) group [11,3].

A natural question may now arise: What is the reason for the existence of scenarios in which gravitational degrees of freedom reassemble in a topologically nontrivial fashion, say, into a Schwarzschild black hole, so that the asymptotic condition (64) is met, and the total energy functional (70) becomes a well defined, non-negative quantity [26,27]? The suggestion can be made that the diffeomorphisms controlling such scenarios are restricted to the groups deprived of free subgroups.

Does the action–reaction principle remain its validity for quantum field theories such as quantum electrodynamics? Three obstacles apparently placed in incorporation of this principle into the quantum context are as follows:

• By virtue of vacuum polarization, the charge of a bare particle is no longer constant, but rather a time-varying dynamical quantity whose numerical value is determined by virtual pair screening. It is unlikely that this fluctuating quantity may be taken to be a measure of both variation of the electron state for a given electromagnetic field state and variation of the state of electromagnetic field for a given electron state.

• Heisenberg's uncertainty principles is contrary to bringing a contact interaction into coincidence with exact values of the four-momenta appearing in the local four-momentum balance. In the quantum realm, the four-momentum balance is either nonlocal or fuzzy.

• The rearrangement of initial degrees of freedom in the quantum picture occurs much differently than in the classical picture. The criterion of stability, Eq. (80), is alien to the quantum regime of evolution because any world line passing through the chosen end points – and not just the world line which renders the action extremal – contributes to the Feynman path integral. Therefore, the instability is of little, if at all, significance for the quantum rearrangement.

However, it would be very strange if the Nature does reject the quantum utility of the principle which is so useful at the classical level.

References

[1] M. Planck, Phys. Z. 9 (1908) 828–830.
[2] E. Noether, Nachr. Kön. Ges. Der Wiss. Göttingen. Math.-Phys. Kl. 2 (1918) 235–258.
[3] B. Kosyakov, Introduction to the Classical Theory of Particles and Fields, Springer, Berlin, 2007.
[4] A.O. Barut, Phys. Rev. D 10 (1974) 3335–3336.
[5] P.A.M. Dirac, Proc. Roy. Soc. A 167 (1938) 148–169.
[6] F. Rohrlich, Classical Charged Particles, Addison–Wesley, Reading, 1965. second ed. 1990, third edition World Scientific, Singapore, 2007.
[7] C. Teitelboim, Phys. Rev. D 1 (1970) 1572–1582.
[8] B.P. Kosyakov, Internat. J. Modern Phys. A 20 (2005) 2459–2464.
[9] S.K. Wong, Nuovo Cimento A 65 (1970) 689–694.
[10] A. Balachandran, S. Borchardt, A. Stern, Phys. Rev. D 17 (1978) 3247–3256.
[11] B.P. Kosyakov, Phys. Rev. D 57 (1998) 5032–5048.
[12] B.P. Kosyakov, Theoret. Math. Phys. 87 (1991) 632–635.
[13] B.P. Kosyakov, Sov. Phys.-Usp. 35 (1992) 135–142.
[14] B.P. Kosyakov, Phys. Lett. B 312 (1993) 471–476.
[15] B.P. Kosyakov, Theoret. Math. Phys. 99 (1994) 409–421.
[16] A. Schmidt-May, M. von Strauss, J. Phys. A 49 (2016) 183001.
[17] K. Schwarzschild, Sitzber. Kön. Preuss. Akad. Wiss. Berlin. Math.-Phys. Kl. 7 (1916) 189–196.
[18] R.W. Fuller, J.A. Wheeler, Phys. Rev. 128 (1962) 919–929.
[19] C. Fronsdal, Phys. Rev. 116 (1959) 778–781.
[20] M.D. Kruskal, Phys. Rev. 119 (1960) 1743–1745.
[21] B.P. Kosyakov, Found. Phys. 38 (2008) 678–694.
[22] R. Arnowitt, S. Deser, C.W. Misner, Phys. Rev. 117 (1960) 1595–1602.
[23] R. Arnowitt, S. Deser, C.W. Misner, Phys. Rev. 118 (1960) 1100–1104.
[24] R. Arnowitt, S. Deser, C.W. Misner, Phys. Rev. 122 (1961) 997–1006.
[25] T. Regge, C. Teitelboim, Ann. Phys. 88 (1974) 286–318.
[26] R. Schoen, S.-T. Yau, Comm. Math. Phys. 65 (1979) 45–76.
[27] E. Witten, Comm. Math. Phys. 80 (1981) 381–402.
[28] V.I. Denisov, V.O. Solov'ev, Theor. Math. Phys. 56 (1983) 832–841.
[29] S. Banach, A. Tarski, Fund. Math. 6 (1924) 244–277.
[30] F. Hausdorff, Math. Ann. 75 (1914) 428–434.
[31] M. Heusler, Black Hole Uniqueness Theorems, Cambridge University Press, Cambridge, 1996.
[32] Chruściel P.T., Black Holes, in: Springer Lecture Notes in Physics, vol. 604, 2002, p. 61.
[33] H. Umezawa, Nuovo Cimento A 40 (1965) 450–475.

[34] R. Penrose, Phys. Rev. Lett. 14 (1965) 57–59.
[35] S.W. Hawking, Proc. R. Soc. A 300 (1967) 187–201.
[36] S.W. Hawking, R. Penrose, Proc. R. Soc. A 314 (1970) 529–548.
[37] S. Wagon, The Banach–Tatski Paradox, Cambridge University Press, Cambridge, 1994.