Self-dual electromagnetic fields

Andrew E. Chubykalo[a] and Augusto Espinoza[b]
Escuela de Física, Universidad Autónoma de Zacatecas, Apartado Postal C-580, Zacatecas 98068, Zacatecas, Mexico

B. P. Kosyakov[c]
Russian Federal Nuclear Center, Sarov, 607190 Nizhnii Novgorod Region, Russia

(Received 12 January 2010; accepted 15 March 2010)

We demonstrate the utility of self-dual fields in electrodynamics. Stable configurations of free electromagnetic fields can be represented as superpositions of standing waves, each possessing zero Poynting vector and zero orbital angular momentum. The standing waves are themselves superpositions of self-dual and anti-self-dual solutions. The idea of self-duality provides additional insights into the geometrical and spectral properties of stable electromagnetic configurations, such as those responsible for the formation of ball lightning.
[DOI: 10.1119/1.3379299]

Self-dual solutions to the Yang–Mills equations were first discovered by Polyakov and coauthors in 1975.[1] Since then, these solutions, known popularly as "instantons," have gained recognition among experts in gauge field theory and mathematical physics. Furthermore, the idea of self-duality was found to be of much significance in many problems of algebraic geometry.

To the best of our knowledge, self-dual configurations have not been considered in classical electrodynamics. It is instructive to consider self-dual fields in this simpler context. Our analysis, apart from its pedagogical value, will show the utility of this concept in problems of experimental interest, notably in ball lightning for which a stable quasilocalized configuration of the free electromagnetic field[2] is likely to be relevant. We will obtain a field configuration similar to that found in Ref. 2 but follow an alternative route based on the idea of self-duality.

There are several reasons for considering self-dual fields in classical electrodynamics: Self-dual solutions are readily calculated and possess trivial energy-momentum properties, and the desired free field configurations are obtainable as superpositions of self-dual and anti-self-dual constituents so that the resulting spectral properties may be easily controlled.

To simplify our notation as much as possible, we choose the Gaussian system of units and set the speed of light equal to unity.

An electromagnetic field is self-dual/anti-self-dual if

$$i\mathbf{E} = \pm\mathbf{B}. \tag{1}$$

Consider free electromagnetic fields governed by the homogeneous Maxwell equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2}$$

$$\nabla \cdot \mathbf{E} = 0, \tag{3}$$

$$\nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t}, \tag{4}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{5}$$

Let some field configuration be self-dual. If this field obeys Eqs. (4) and (5), it automatically obeys Eqs. (2) and (3). Because Maxwell's equations are linear, any superposition of self-dual and anti-self-dual solutions is a further solution.

The condition that a field configuration is self-dual is not invariant under the parity transformation $\mathbf{r}\to-\mathbf{r}$ because of the opposite parity properties of the electric and magnetic field; the mirror-image configuration is anti-self-dual. As will become clear, the physically relevant configurations are represented by a sum of self-dual and anti-self-dual solutions, which is invariant under the parity transformation.

Let us express the electric field intensity $\mathbf{E}$ and the magnetic induction $\mathbf{B}$ in terms of scalar and vector potentials ϕ and $\mathbf{A}$. Then the self-duality condition (1) becomes

$$\mp i\left(\nabla\phi + \frac{\partial \mathbf{A}}{\partial t}\right) = \nabla \times \mathbf{A}. \tag{6}$$

If we fix the gauge $\phi=0$, then Eq. (6) reduces to

$$\mp i\frac{\partial \mathbf{A}}{\partial t} = \nabla \times \mathbf{A}. \tag{7}$$

Because the self-duality condition (7) is a linear first-order partial differential equation, it is simpler to solve than the second-order equations that result from Maxwell's Eqs. (2)–(5).

A remarkable property of self-dual configurations is that they carry zero energy and momentum. This property can be verified by applying the self-duality condition (1) to the expressions for the energy density $\varepsilon=(1/8\pi)(\mathbf{E}^2+\mathbf{B}^2)$ and the Poynting vector $\mathbf{S}=(1/4\pi)(\mathbf{E}\times\mathbf{B})$.

Note that given an antisymmetric field $F_{\mu\nu}$ in Minkowski space, the self-duality condition can be expressed as

$${}^*F_{\mu\nu} = \pm iF_{\mu\nu}, \tag{8}$$

where the Hodge dual field ${}^*F_{\mu\nu}$ is defined by ${}^*F_{\mu\nu}=\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$. Equation (8) is identical to Eq. (1) because $\mathbf{E}$ and $\mathbf{B}$ are expressed in terms of $F_{\mu\nu}$ as $E_i=F_{0i}$ and $B_i=-\frac{1}{2}\epsilon_{ijk}F_{jk}$.

If the Bianchi identity

$$\partial_\mu {}^*F^{\mu\nu} = 0 \tag{9}$$

is compared with the equations of motion for a free electromagnetic field,

$$\partial_\mu F^{\mu\nu} = 0, \tag{10}$$

it becomes apparent that if $F_{\mu\nu}$ obeys Eqs. (8) and (9), then $F_{\mu\nu}$ automatically obeys Eq. (10).

Self-dual configurations possess trivial energy-momentum contents. The stress-energy tensor

$$\Theta_{\mu\nu} = \frac{1}{4\pi}\left(F_\mu{}^\lambda F_{\lambda\nu} + \frac{\eta_{\mu\nu}}{4} F^{\alpha\beta} F_{\alpha\beta}\right) \tag{11}$$

can be brought into the form

$$\Theta_{\mu\nu} = \frac{1}{8\pi}(F_\mu{}^\lambda F_{\lambda\nu} + {}^*F_\mu{}^\lambda\, {}^*F_{\lambda\nu}) \tag{12}$$

(the proof is simple; see, for example, Ref. 3, Problem 5.2.8), and thus

$$\Theta_{\mu\nu} = \frac{1}{8\pi}(F_\mu{}^\lambda + i\, {}^*F_\mu{}^\lambda)(F_{\lambda\nu} - i\, {}^*F_{\lambda\nu}). \tag{13}$$

If ${}^*F_{\mu\nu} = \pm iF_{\mu\nu}$, then $\Theta_{\mu\nu}=0$. However, if we take a superposition of self-dual and anti-self-dual fields, we have $\Theta_{\mu\nu} \neq 0$ because the contribution of the mixed terms to the stress-energy tensor need not vanish.

A field with nontrivial energy-momentum properties made up of two zero energy-momentum modes is analogous to a massive particle that can disintegrate into two zero-mass particles, say, $\pi^0 \to 2\gamma$. Let p^μ be the four-momentum of a pion, $p^2=m^2$, which can decay into two photons with four-momenta k_1^μ and k_2^μ. In a particular Lorentz frame, $k_1^\mu = (e_1, e_1\mathbf{n}_1)$ and $k_2^\mu = (e_2, e_2\mathbf{n}_2)$, where e_1 and e_2 are energies of those photons and $\mathbf{n}_1$ and $\mathbf{n}_2$ are spatial unit vectors so that $k_1^2=0$ and $k_2^2=0$. Because $p^\mu = k_1^\mu + k_2^\mu$, we have

$$p^2 = m^2 = (k_1 + k_2)^2 = 2k_1 \cdot k_2 = 2e_1 e_2(1 - \mathbf{n}_1 \cdot \mathbf{n}_2) \geq 0. \tag{14}$$

The responsibility for making m nonzero rests with the mixed term $2k_1 \cdot k_2$.

The factor of i in the definition of self-duality (8) is unavoidable if we work in Minkowski space where ${}^{**}F = -F$. Therefore, in Minkowski space self-dual field configurations contain complex-valued fields. In contrast, in Euclidean space the factor of i is absent and self-dual fields can be real. We note that fundamental physical laws are usually formulated in the form of differential equations with real coefficients. However, this formulation does not necessarily imply that every solution to such equations is real. The only *a priori* constraint, stemming from the fact that the coefficients of the differential equations are real, is that each complex solution is accompanied by a complex-conjugate solution; complex fields occur only as pairs of complex-conjugate solutions. The natural way for constructing observable configurations from complex solutions to Maxwell's equations is to add complex-conjugate solutions, in particular, self-dual and anti-self-dual solutions.

Our main interest here is with solutions corresponding to standing waves which are due to the interplay of converging and diverging microwaves.[6] Therefore, the spacetime behavior of the desired solutions factorizes into t- and $\mathbf{x}$-dependent factors,

$$\mathbf{A}(t,\mathbf{x}) = \alpha(t)\mathbf{a}(\mathbf{x}). \tag{15}$$

We substitute Eq. (15) into Eq. (7) to obtain

$$\frac{\dot{\alpha}}{\alpha} = \pm ik, \tag{16}$$

where k is an arbitrary integration constant. It follows that

$$\alpha_k^{(\pm)}(t) = a_k^{(\pm)} \exp(\pm ikt), \tag{17}$$

where $a_k^{(\pm)}$ are more integration constants. Equation (7) becomes

$$k\mathbf{a} = \boldsymbol{\nabla} \times \mathbf{a}. \tag{18}$$

Among solutions to Eq. (18), there are waves with the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ parallel to each other and both perpendicular to the propagating vector $\mathbf{k}$ such as

$$\mathbf{k} = k(0,0,1), \tag{19}$$

$$\mathbf{A}_k^{(\pm)} = a_k^{(\pm)}(\sin kz, \cos kz, 0)\exp(\pm ikt), \tag{20}$$

$$\mathbf{E}_k^{(\pm)} = \mp ika_k^{(\pm)}(\sin kz, \cos kz, 0)\exp(\pm ikt), \tag{21}$$

$$\mathbf{B}_k^{(\pm)} = ka_k^{(\pm)}(\sin kz, \cos kz, 0)\exp(\pm ikt). \tag{22}$$

These solutions are complex conjugates of one another. They describe transverse waves with $\mathbf{E}$ parallel to $\mathbf{B}$ so that $\mathbf{S} = (1/4\pi)(\mathbf{E}\times\mathbf{B}) = 0$. $\mathbf{A}^{(+)}$ is self-dual and $\mathbf{A}^{(-)}$ is anti-self-dual. Both are characterized by vanishing energy density $\varepsilon = (1/8\pi)(\mathbf{E}^2 + \mathbf{B}^2) = 0$. The properties $\mathbf{S}=0$ and $\varepsilon=0$ are frame-independent because for self-dual fields, $\Theta_{\mu\nu}=0$ holds in any Lorentz frame. If we let $a_k^{(+)} = a_k^{(-)} = (1/2)A$ and add the + and − modes, we arrive at the real solution given in Ref. 7,

$$\mathbf{A} = A(\sin kz, \cos kz, 0)\cos kt, \tag{23}$$

$$\mathbf{E} = kA(\sin kz, \cos kz, 0)\sin kt, \tag{24}$$

$$\mathbf{B} = kA(\sin kz, \cos kz, 0)\cos kt. \tag{25}$$

This solution corresponds to transverse standing waves perpendicular to the z-axis. In the present Lorentz frame, $\mathbf{E}$ is parallel to $\mathbf{B}$, and hence $\mathbf{S}=0$ (in other frames, this equality is not the case). Note that the energy density is finite, $\varepsilon = (1/8\pi)k^2A^2$.

It is easy to visualize the field configuration (24) and (25). Consider a vector $\mathbf{E}_0$ perpendicular to the z-axis, which moves uniformly at the velocity $|\mathbf{E}_0|k$ along the z-axis and rotates with angular frequency k (see Fig. 1). The ruled surface swept out by $\mathbf{E}_0$, the *helicoid*, provides a snapshot of the electric field $\mathbf{E}$ described by Eq. (24). This helicoid rescales in time following the overall factor $\sin kt$. The same is true for the magnetic field $\mathbf{B}$ whose oscillatory rescaling is

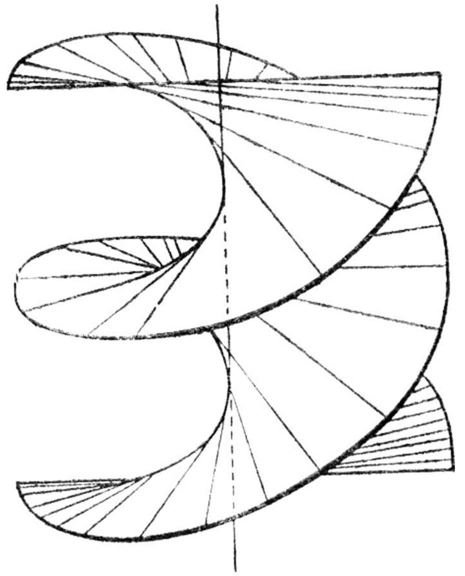

Fig. 1. A snapshot of the field configuration given by Eq. (24).

displaced in phase by $\frac{1}{2}\pi$ with respect to the rescaling of $\mathbf{E}$.

The application of the curl operator to Eq. (18) gives

$$k\mathbf{B} = \nabla \times \mathbf{B}, \tag{26}$$

where $\mathbf{B} = \nabla \times \mathbf{A}$ is the magnetic field associated with the vector potential $\mathbf{A}$. Equation (26) governs the relaxed states of a slightly resistive turbulent plasma.[8]

We now look for axially symmetric solutions to Eq. (18). In spherical coordinates,

$$\mathbf{a} = \mathcal{A}_r \mathbf{e}_r + \mathcal{A}_\vartheta \mathbf{e}_\vartheta + \mathcal{A}_\varphi \mathbf{e}_\varphi, \tag{27}$$

and

$$\begin{aligned}\nabla \times \mathbf{a} = &\frac{1}{r}\left(\frac{\partial \mathcal{A}_\varphi}{\partial \vartheta} - \frac{1}{\sin\vartheta}\frac{\partial \mathcal{A}_\vartheta}{\partial \varphi} + \cot\vartheta \mathcal{A}_\varphi\right)\mathbf{e}_r \\ &+ \left(\frac{1}{r\sin\vartheta}\frac{\partial \mathcal{A}_r}{\partial \varphi} - \frac{\partial \mathcal{A}_\varphi}{\partial r} - \frac{1}{r}\mathcal{A}_\varphi\right)\mathbf{e}_\vartheta \\ &+ \left(\frac{\partial \mathcal{A}_\vartheta}{\partial r} - \frac{1}{r}\frac{\partial \mathcal{A}_r}{\partial \vartheta} + \frac{1}{r}\mathcal{A}_\vartheta\right)\mathbf{e}_\varphi.\end{aligned} \tag{28}$$

By axial symmetry, the coefficients $\mathcal{A}_\mathbf{r}$, $\mathcal{A}_\vartheta$, and $\mathcal{A}_\varphi$ are independent of φ, and thus Eq. (18) becomes

$$\left(\frac{\partial}{\partial \vartheta} + \cot\vartheta\right)\mathcal{A}_\varphi = kr\mathcal{A}_r, \tag{29}$$

$$D_r \mathcal{A}_\varphi = -k\mathcal{A}_\vartheta, \tag{30}$$

$$D_r \mathcal{A}_\vartheta - \frac{1}{r}\frac{\partial \mathcal{A}_r}{\partial \vartheta} = k\mathcal{A}_\varphi, \tag{31}$$

where differential operator D_r is defined as

$$D_r = \frac{\partial}{\partial r} + \frac{1}{r}. \tag{32}$$

We assume the simplest ϑ-dependence of the vector potential compatible with Eqs. (29) and (31),

$$\mathbf{a} = \cos\vartheta R_r(r)\mathbf{e}_r + \sin\vartheta R_\vartheta(r)\mathbf{e}_\vartheta + \sin\vartheta R_\varphi(r)\mathbf{e}_\varphi. \tag{33}$$

With this ansatz, we combine Eqs. (29)–(31) to obtain

$$\left(D_r D_r - \frac{2}{r^2} + k^2\right)R_\varphi = 0. \tag{34}$$

If we take into account that $D_r D_r R_\varphi = r^{-2}(r^2 R'_\varphi)'$, where the prime stands for the derivative with respect to r, we find that Eq. (34) is identical to the radial part of the Schrödinger equation for a free particle (see, for example, Ref. 9, Sec. 33),

$$R''_{k\ell} + \frac{2}{r}R'_{k\ell} + \left[k^2 - \frac{\ell(\ell+1)}{r^2}\right]R_{k\ell} = 0, \tag{35}$$

provided that $\ell = 1$. It is well known that the regular solution of Eq. (35) is proportional to the spherical Bessel functions $j_\ell(kr) = (\pi/2r)^{1/2}J_{\ell+1/2}(kr)$.[9] We let $\ell = 1$ and obtain

$$R_\varphi(r) = \frac{\lambda_k}{r} = \mathcal{C}_k j_1(kr) = -\frac{\mathcal{C}_k}{k^2}\left(\frac{\sin kr}{r}\right)'. \tag{36}$$

Here, $\lambda_k = \lambda_k(r)$ is defined by Eq. (36) and $\mathcal{C}_k$ is an arbitrary constant. The desired self-dual/anti-self-dual solution is

$$\mathbf{A}_k^{(\pm)} = \frac{1}{r}\left(\frac{2\lambda_k}{kr}\cos\vartheta \mathbf{e}_r - \frac{\lambda'_k}{k}\sin\vartheta \mathbf{e}_\vartheta + \lambda_k \sin\vartheta \mathbf{e}_\varphi\right) \times \exp(\pm ikt), \tag{37a}$$

$$\mathbf{E}_k^{(\pm)} = \mp ik\mathbf{A}^{(\pm)}, \tag{37b}$$

$$\mathbf{B}_k^{(\pm)} = k\mathbf{A}_k^{(\pm)}. \tag{37c}$$

We add one-half the + and − modes to obtain

$$\mathbf{A}_k = \frac{1}{r}\left(\cos\vartheta \frac{2\lambda_k}{kr}\mathbf{e}_r - \sin\vartheta \frac{\lambda'_k}{k}\mathbf{e}_\vartheta + \sin\vartheta \lambda_k \mathbf{e}_\varphi\right)\cos kt, \tag{38a}$$

$$\mathbf{E}_k = k\mathbf{A}_k \tan kt, \tag{38b}$$

$$\mathbf{B}_k = k\mathbf{A}_k. \tag{38c}$$

This solution describes an axially symmetric configuration of standing electromagnetic waves with $\mathbf{E}_k$ parallel to $\mathbf{B}_k$.

The form of this solution is simple, but its visualization is not illuminating, and thus the analytical result may give more insight into the nature of this solution. All the field lines are closed. Every field line is confined in a spherical shell [whose inner and outer radii are determined by the relevant adjacent roots of $j_1(kr) = 0$] in the sense that the radial components of the fields $\mathbf{E}$ and $\mathbf{B}$ tangent to this line go to zero as the line approaches either boundary of the shell.

The field configuration (38) possesses finite energy density, $\varepsilon = (1/8\pi)k^2\mathbf{A}_k^2$, but its Poynting vector is zero. Therefore, energy is not carried along this configuration. Because $\mathbf{E}_k$

and $\mathbf{B}_k$ decrease as $1/r$, this standing wave is characterized by infinite energy. In particular, an infinitely large amount of energy is found outside any sphere of finite radius. Such superpositions are called quasilocalized.[10]

Although configuration (38) is not invariant under the full rotation group, it exhibits zero orbital angular momentum,

$$\mathbf{L} = \frac{1}{4\pi}\int d^3x\mathbf{x} \times (\mathbf{E}_k \times \mathbf{B}_k) = 0. \tag{39}$$

We see that stable configurations of free electromagnetic fields, both extended and quasilocalized, can be represented as superpositions of standing waves, each individually possessing zero Poynting vector and zero orbital angular momentum. The standing waves are superpositions of self-dual and anti-self-dual fields that are solutions to Eq. (7). Simple examples of self-dual solutions, possessing trivial energy-momentum content, are given by Eqs. (20)–(22) and (37).

Why is the field configuration (38) interesting? This standing wave seems to owe its origin to the interplay of converging and diverging axially symmetric microwaves, which is a prerequisite to the ball lightning formation.[4,5] This wave is the simplest axially symmetric solution to Eq. (26), which is important in the theory of plasma relaxation.[8] Thus, Eqs. (38) show promise for providing the first approximation to a self-consistent background field in the ball lightning interior.

Solution (38) should not be confused with the electromagnetic field emitted by ball lightning. A reasonable strategy for obtaining the ball lightning radiation is to solve the Schrödinger equation with this electromagnetic standing-wave background and find the bremsstrahlung effect due to electron scattering on these standing waves. Unfortunately, this problem is difficult mathematically. Even the behavior of a classical charged particle in this background is involved. A diligent student might try to find the evolution of a charged particle in the simplest background given by Eqs. (24) and (25).

[a)]Electronic mail: achubykalo@yahoo.com.mx
[b)]Electronic mail: drespinozag@yahoo.com.mx
[c)]Electronic mail: kosyakov@vniief.ru

[1] A. A. Belavin, A. M. Polyakov, A. S. Schwartz, and Yu. S. Tyupkin, "Pseudoparticle solutions of the Yang–Mills equations," Phys. Lett. B **59**, 85–87 (1975).
[2] A. E. Chubykalo and A. Espinoza, "Unusual formations of the free electromagnetic field in vacuum," J. Phys. A **35**, 8043–8053 (2002).
[3] B. Kosyakov, *Introduction to the Classical Theory of Particles and Fields* (Springer, Heidelberg, 2007).
[4] P. L. Kapitza, "On the nature of ball lightning," Dokl. Akad. Nauk SSSR **101**, 245–248 (1955) (in Russian).
[5] M. Stenhoff, *Ball Lightning: An Unsolved Problem in Atmospheric Physics* (Kluwer, New York, 1999).
[6] This conjectured steady-state condition for the field configuration is in the same spirit as that proposed by Kapitza (Ref. 4) as an essential prerequisite to ball lightning formation. Interested readers might find Ref. 5 to be a useful guide to ball lightning with an extensive bibliography of about 2400 references.
[7] C. Chu and T. Ohkawa, "Transverse electromagnetic waves with $\mathbf{E}\|\mathbf{B}$," Phys. Rev. Lett. **48**, 837–838 (1982).
[8] J. B. Taylor, "Relaxation and magnetic reconnection in plasmas," Rev. Mod. Phys. **58**, 741–763 (1986).
[9] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon, Oxford, 1963).
[10] The expert reader will recognize that the free particle wave function with a definite angular orbital momentum, whose radial part obeys Eq. (35), is a non-normalizable solution to the free particle Schrödinger equation, which is another way of stating that the standing wave (38) is characterized by infinite energy. Motivated by the rigorous treatment of scattering problems in which normalizable wave packets are taken from the outset, we should construct a physically achievable, localized field configuration carrying a range of k's by summing over different quasilocalized modes that are exact solutions to Eq. (18).