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On Some Unusual Properties of Wave Solutions of Free Maxwell Equations

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Sobre algunas propiedades inusuales de las ecuaciones libres de Maxwell

Resumen: Se descubren algunas propiedades inusuales de las soluciones de las llamadas ecuaciones *libres* de Maxwell. Mostramos la existencia de soluciones que representan las ondas electromagnéticas en el vacío para los cuales el vector de Poynting no coincide con el vector de Umov. Se presentan soluciones que corresponden a ondas magnéticas estacionarias de una configuración inusual en el vacío, que describen en el vacio formaciones estables anulares y esféricas de campo. Se demuestra que en el vacío, de acuerdo a las soluciones obtenidas el campo eléctrico **E** puede ser un vector polar así como un vector axial; y el campo magnético **B**, en su turno, puede ser un vector axial así como también un vector polar. Se muestra que tales soluciones existen cuando los vectores **E** y **B**, no son vectores polares ni axiales. Además, estas soluciones corresponden a ondas electromagnéticas que no transfieren energía ni momentos en cualquier punto del vacío.

Palabras clave: vector de Poynting, vector de Umov, vector axial, vector polar, soluciones ondulatorias.

Abstract: Some unusual properties of solutions of so called free Maxwell equations are discovered. We show the existence of solutions that represent electromagnetic waves in a vacuum for which the Poynting vector does not coincide with the Umov vector. We show solutions which correspond to standing magnetic waves of an unusual configuration in a vacuum; solutions describing spherical and ring-like stable field formations in vacuum. It is shown that in a vacuum, according to the solutions obtained, the electric field E can be a polar vector as well as an axial vector, and the magnetic field B, in turn, can be an axial vector as well as a polar vector. It is also shown that such solutions exist when the vectors E and B are neither polar vectors nor axial vectors. Furthermore, these solutions correspond to electromagnetic waves which transfer neither energy nor momentum at any point in a vacuum.

Key words: Poynting vector, Umov vector, axial vector, polar vector, wave solutions.

Introduction

Students of physics mainly suppose that solutions of *free* Maxwell equations (FME) (*i.e.* Maxwell equations without charges and currents) do not lead to anything especial surprising. However, a set of solutions of FME exists which refute this almost generally accepted opinion. It should also be stated that the existence of such solutions almost is not in the picture in modern textbooks.[1]

It is well known that from the system of complete Maxwell equations (CME) (*i.e.* equations with non-zero density of a charge and non-zero density of a current of conductivity) it follows that any solution **E** and **B** of this system must be **E**-*polar vector* and **B**-*axial vector* in the microscopic consideration. One can satisfy oneself that this

1. Truly, in the textbook by M. Born and E. Wolf (1999) the general solution of FME was adduced. This general solution was obtained by G. M. Mie (1980). Of course, this solution formally encompasses all possible particular cases. Our article is devoted to obtaining and investigating those unusual particular solutions of FME mentioned in the Introduction.

fact directly follows from equation div **E** = 4πϱ and from other equations of CME. Actually, seeing that ϱ is a scalar with respect to the spatial inversion transformation for coordinates and the operator "div" changes its sign, the vector **E** is a polar vector because it behaves as $\mathbf{E} \rightarrow \mathbf{E}' = -\mathbf{E}$.

If one follows this claim, then it is obvious that so called *free* Maxwell equations (FME)

$$\operatorname{div} \mathbf{E} = 0, \tag{1}$$

$$\operatorname{rot} \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \tag{2}$$

$$\operatorname{div} \mathbf{B} = 0, \tag{3}$$

$$\operatorname{rot} \mathbf{B} = -\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}, \tag{4}$$

must correspondingly have only solutions of this type.

However, if we consider FME from a purely formal, mathematical point of view, then this condition (**E**-*polar vector* and **B**-*axial vector*) is not obligatory. The point is that one of the authors of the present work argued (Chubykalo *et al*., 1998) of Gauss' law to constructing CME leads to realizing the fact that FME and, correspondingly, so called *free* electric and magnetic fields are not *a consequence* of CME: one just may *postulate* them.

Thus the present work is devoted to the research of some unusual theoretical results connected with certain solutions of FME. We show below that such *nonstandard* solutions of FME exist, when the electric field **E** can be a polar vector as well as an axial vector, and the magnetic field **B** in turn can be an axial vector as well as a polar vector. Of course, we omit the consideration of the trivial case when one can make the following interchange in FME: $\mathbf{E} \rightarrow -\mathbf{B}'$, $\mathbf{B} \rightarrow \mathbf{E}'$. Obviously, the axial vector $\mathbf{E}'$ and the polar vector $\mathbf{B}'$ are solutions of FME too. And it will be shown also that such a solution exists when the vectors **E** and **B** are neither polar vectors nor axial vectors. Generally speaking, if one considers free Maxwell equations without taking into account a history of their origin, one can formally construct solutions that are not subject to polarity conditions. In Section 1 we show a way of constructing these solutions.

We show that even solutions with a standard polarity lead to unusual results such as ball-like and ring-like stable free field formations in vacuum. We also show that in such solutions, absolute values of the Poynting vector $\mathbf{S} = \frac{c}{4\pi}(\mathbf{E}\times\mathbf{B})$ and the Umov vector (energy-flux) vector $\mathbf{U} = wc\mathbf{n}$ do not coincide! Here **n** is a unit vector along the direction of propagation of the energy, *i.e*. along the direction of the Poynting vector.

1. Unusual Solutions of FME with Different Polarities

One can represent the solutions of (1-4) as follows:

$$\mathbf{E} = [C_1\mathbf{e}(r) + C_2\mathbf{b}(r)]\sin(\Omega t) \text{ and } \mathbf{B} = [C_1\mathbf{b}(r) + C_2\mathbf{e}(r)]\cos(\Omega t), \tag{5}$$

where C_1, C_2 and Ω are *constant* and where vectors **e** and **b** are solution of the system

$$\operatorname{rot}\mathbf{e} = \frac{\Omega}{c}\mathbf{b}; \quad \operatorname{rot}\mathbf{b} = \frac{\Omega}{c}\mathbf{e} \tag{6}$$

such that **e** is a polar vector and **b** is an axial vector (*c* is the velocity of light in vacuum). This solution expressed by components (cartesian and spherical ones) is:

$$\mathbf{e} = D\left\{\frac{-\alpha\Omega y}{cr^3}, \frac{\alpha\Omega x}{cr^3}, 0\right\} = \frac{\Omega\alpha\sin\theta}{cr^2}D\hat{\mathbf{e}}_\varphi \tag{7}$$

and

$$\mathbf{b} = D\left\{\frac{\beta xz}{r^5}, \frac{\beta yz}{r^5}, \frac{2\alpha}{r^3} - \frac{\beta(x^2+y^2)}{r^5}\right\} = \frac{2\alpha\cos\theta}{r^3}D\hat{\mathbf{e}}_r + \frac{(\beta-2\alpha)\sin\theta}{r^3}D\hat{\mathbf{e}}_\theta \tag{8}$$

where D is a dimension constant $[D] = M^{1/2}L^{5/2}T^{-1}$;

$$\beta = 3\alpha - \frac{\Omega^2}{c^2}r^2\sin\left(\frac{\Omega}{c}r\right);$$

$$\alpha = -\frac{\Omega}{c}r\cos\left(\frac{\Omega}{c}r\right) + \sin\left(\frac{\Omega}{c}r\right); \quad r = (x^2+y^2+z^2)^{1/2};$$

and here $\hat{\mathbf{e}}_\varphi$, $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\theta$, are *orts* of the spherical coordinate system. From the whole class of solutions of the system (6) we have chosen the most simple non-trivial one. We adduce the detailed way of obtaining this solution in Section 3.

Let us consider the solutions (5) for some different constants C_1 and C_2.

Following to I.E.Tamm (1957), we accept that identification of the Poynting vector with the energy flux in a given point leads to the equality of the velocity of transmission of energy and the velocity of the carriers of this energy, that is, the electromagnetic waves. In other words, the existence of the fact that, in every point in a given area of a space, the Poynting vector is not *zero*, attests that in this area the energy transferring process takes place.

2.1. Case T_1: Transferring Energy Wave Solution with an Usual Polarity

Let $C_1 = 1$ and $C_2 = 0$. In this case from Eqs. (5) we have:

$$\mathbf{E}_{T1} = \mathbf{e}(\mathbf{r})\sin(\Omega t) \text{ and } \mathbf{B}_{T1} = \mathbf{b}(\mathbf{r})\cos(\Omega t) \tag{9}$$

In this case

$$\mathbf{E}_{T1}\cdot\mathbf{B}_{T1}=0 \text{ and } \mathbf{S}_{T1}=\frac{c}{4\pi}\left(\mathbf{E}_{T1}\times\mathbf{B}_{T1}\right)=\frac{c}{8\pi}(\mathbf{e}\times\mathbf{b})\sin(2\Omega t)\neq 0 \quad (10)$$

$$\mathrm{w}_{T1}=\frac{\mathbf{E}_{T1}^2+\mathbf{B}_{T1}^2}{8\pi}=\frac{1}{8\pi}\left[|\mathbf{e}|^2\sin^2(\Omega t)+|\mathbf{b}|^2\cos^2(\Omega t)\right], \quad (11)$$

where w_{T_1} is the density of energy, $\mathbf{S}_{T_1}$ is the energy-flux vector, index T_1 means that this solution corresponds to the case (T_1). Note that the energy distribution in this field has a central symmetry as well as an axial one with respect to the origin of coordinates and *Z*-axis. If we write the solutions (5) in the spherical system of coordinates

$$\mathbf{E}_{T1}=A_1(\mathrm{r},\theta)\left\{\cos\left(\Omega\left(\frac{\mathrm{r}}{\mathrm{c}}+t\right)-\xi(\mathrm{r})\right)-\cos\left(\Omega\left(\frac{\mathrm{r}}{\mathrm{c}}-t\right)-\xi(\mathrm{r})\right)\right\}\hat{\mathbf{e}}_\varphi \quad (12)$$

and

$$\mathbf{B}_{T1}=\frac{\mathrm{D}\Omega^2\sin\theta}{2c^2\mathrm{r}}\left\{\sin\left(\Omega\left(\frac{\mathrm{r}}{\mathrm{c}}+t\right)\right)+\sin\left(\Omega\left(\frac{\mathrm{r}}{\mathrm{c}}-t\right)\right)\right\}\hat{\mathbf{e}}_\theta-$$
$$\left(A_2(\mathrm{r},\theta)\hat{\mathbf{e}}_\mathrm{r}+\frac{c}{\Omega\mathrm{r}}A_1(\mathrm{r},\theta)\hat{\mathbf{e}}_\theta\right)$$
$$\left\{\sin\left(\Omega\left(\frac{\mathrm{r}}{\mathrm{c}}+t\right)-\xi(\mathrm{r})\right)+\sin\left(\Omega\left(\frac{\mathrm{r}}{\mathrm{c}}-t\right)-\xi(\mathrm{r})\right)\right\}, \quad (13)$$

where

$$A_1(\mathrm{r},\theta)=\frac{\mathrm{D}\Omega\sin\theta}{2c\mathrm{r}^2}\sqrt{1+\left(\frac{\Omega\mathrm{r}}{\mathrm{c}}\right)^2}\;;\; A_2(\mathrm{r},\theta)=\frac{\mathrm{D}\cos\theta}{\mathrm{r}^3}\sqrt{1+\left(\frac{\Omega\mathrm{r}}{\mathrm{c}}\right)^2}$$
$$\xi(\mathrm{r})=\arctan\left(\frac{\Omega\mathrm{r}}{\mathrm{c}}\right),$$

one can see that they are a superposition of intricate electromagnetic waves with amplitudes and phases depending on r, θ and spreading along opposite directions from the origin of coordinates to infinity and conversely.

We see, besides, that these waves which are usual in the sense of *polarity* nevertheless are unusual enough. For example, let us consider the behavior of the wave fields (12) and (13) along the *Z*-axis. In the spherical coordinates it will mean that $\theta = 0$. The electric field (12) along this axis is zero according to the value of $A_1(\mathrm{r},\theta)$, and the magnetic field (13) becomes:

$$\mathbf{B}_{T1}=\frac{\mathrm{D}\sqrt{1+\left(\frac{\Omega\mathrm{r}}{\mathrm{c}}\right)}}{\mathrm{r}^3}\hat{\mathbf{e}}_\mathrm{r}\left\{\sin\left(\Omega\left(\frac{\mathrm{r}}{\mathrm{c}}+t\right)-\xi(\mathrm{r})\right)+\sin\left(\Omega\left(\frac{\mathrm{r}}{\mathrm{c}}-t\right)-\xi(\mathrm{r})\right)\right\}. \quad (14)$$

It is clear that Eq.(14) represents a *standing* "longitudinal"[2] *magnetic* (not *electro*magnetic!) wave along the *Z*-axis (we call it "longitudinal" because vector **B** vibrates along *Z*). In order to prove that (14) is really a *standing* wave we just have to prove that solutions r of the equation $\mathbf{B}_{T_1} = 0$, which define nodal points of a wave, do not depend on time *t*, i.e. $\mathrm{r} \neq \mathrm{r}(t)$. It is easy to show that these solutions are also solutions of the equation

$$\tan\left(\frac{\Omega\mathrm{r}}{\mathrm{c}}\right)=\frac{\Omega\mathrm{r}}{\mathrm{c}}.$$

Hence, one can see that r does not depend on *t* and, consequently, Eq. (14) represents the *standing* longitudinal wave of a magnetic field.

Vector $\mathbf{B}_{T_1}$ disappears with $\mathrm{r}\to\infty$ and in the point r = 0:

$$\lim_{\mathrm{r}\to 0}|\mathbf{B}_{T1}|=\frac{2\mathrm{D}\Omega^3\cos(\Omega t)}{3c^3}. \quad (15)$$

Aside from the unusual properties of the solution T_1 cited, we adduce another curious property. In our recent work (Chubykalo and Espinoza, 2002) it is shown that for the solution T_1 (Eqs. 12 and 13) electromagnetic energy within spheres of the radii R which are solutions of the equations

$$\tan\left(\frac{\Omega\mathrm{R}}{\mathrm{c}}\right)=\frac{\Omega\mathrm{R}}{\mathrm{c}}$$

or

$$\tan\left(\frac{\Omega\mathrm{R}}{\mathrm{c}}\right)=\frac{\frac{\Omega\mathrm{R}}{\mathrm{c}}}{1-\frac{\Omega^2\mathrm{R}^2}{\mathrm{c}^2}},$$

does not change with time. Let us also direct attention to an interesting fact; at the surfaces of the spheres of the radius R, from the first of these equations only the magnetic field is present, and the electric field at these surfaces does not exist. We call these spheres *magnetic spheres*. In turn there are rings at the plane z = 0 with radii satisfying the second of these equations, where a magnetic field is not present. We call these rings *electric rings*. We emphasize that these surfaces and rings do not deform, do not displace and do not rotate with time in vacuum.

2.2. Case T2: Transferring Energy Wave Solution with an Inverse Polarity

Now we consider the case when in Eqs. (5) $\mathrm{C}_1 = 0$ and $\mathrm{C}_2 = 1$. In this case from Eqs. (5) we have:

$$\mathbf{E}_{T2}=\mathbf{b}(\mathbf{r})\sin(\Omega t) \text{ and } \mathbf{B}_{T2}=\mathbf{e}(\mathbf{r})\cos(\Omega t) \quad (16)$$

2 It is the fact unexpected enough because there is the generally accepted point of view in textbooks that theoretically exclusively *transversal* waves can exist in vacuum.

We see now that this solution is already unusual in the sense of *polarity*: $\mathbf{E}_{T2}$ is an axial vector and $\mathbf{B}_{T2}$ is a polar one! It is easy to show that the electromagnetic wave formed by this field spreads in the opposite direction with respect to the direction of wave spreading (12), (13). Actually

$$\mathbf{E}_{T2}\cdot\mathbf{B}_{T2}=0 \text{ and } \mathbf{S}_{T2}=\frac{c}{4\pi}\left(\mathbf{E}_{T2}\times\mathbf{B}_{T2}\right)=-\frac{c}{8\pi}(\mathbf{e}\times\mathbf{b})\sin(2\Omega t) \quad (17)$$

and the energy density

$$\mathrm{w}_{T2}=\frac{\mathbf{E}_{T2}^2+\mathbf{B}_{T2}^2}{8\pi}=\frac{1}{8\pi}\left[|\mathbf{b}|^2\sin^2(\Omega t)+|\mathbf{e}|^2\cos^2(\Omega t)\right]. \quad (18)$$

Here, however, a certain doubt can arise: can one apply these definitions of the energy density and the energy-flux vector to fields of nonstandard polarity? It easy to show that one can apply these definitions to these nonstandard fields if the mentioned fields are solutions of FME. Indeed, let us multiply both sides of Eq. (2) by $\mathbf{B}_{T2}$ and both sides of Eq. (4) by $\mathbf{E}_{T2}$ (taking into account that $\mathbf{B}_{T2}$ and $\mathbf{E}_{T2}$ are solutions of these equations) and combine the resultant equations. Then we get

$$\frac{1}{c}\mathbf{E}_{T2}\cdot\frac{\partial\mathbf{E}_{T2}}{\partial t}+\frac{1}{c}\mathbf{B}_{T2}\cdot\frac{\partial\mathbf{B}_{T2}}{\partial t}=-\left(\mathbf{B}_{T2}\cdot\mathrm{rot}\mathbf{E}_{T2}-\mathbf{E}_{T2}\cdot\mathrm{rot}\mathbf{B}_{T2}\right). \quad (19)$$

Using the well-known formula of vector analysis, we rewrite this relation in the form

$$\frac{1}{2c}\frac{\partial}{\partial t}\left(\mathbf{E}_{T2}^2+\mathbf{B}_{T2}^2\right)=-\mathrm{div}\left(\mathbf{E}_{T2}\times\mathbf{B}_{T2}\right)$$

or after multiplying this relation by $\frac{c}{4\pi}$

$$\frac{\partial}{\partial t}\left(\frac{\mathbf{E}_{T2}^2+\mathbf{B}_{T2}^2}{8\pi}\right)=-\mathrm{div}\mathbf{S}_{T2}. \quad (20)$$

Now we show that the vector

$$\mathbf{S}_{T2}=\frac{c}{4\pi}\left(\mathbf{E}_{T2}\times\mathbf{B}_{T2}\right) \quad (21)$$

and the relation $\left(\mathbf{E}_{T2}^2+\mathbf{B}_{T2}^2\right)/(8\pi)$ are the energy-flux vector and the energy density of the field (16). First we integrate (20) over a volume and apply Gauss' theorem to the term on the right. Then we obtain[3]

$$\frac{d}{dt}\int\frac{\mathbf{E}_{T2}^2+\mathbf{B}_{T2}^2}{8\pi}dV=-\oint\mathbf{S}_{T2}\cdot d\mathbf{f} \quad (22)$$

If the integral extends over *all* space, then the surface integral vanishes because the field is zero at infinity (one can see this from Eqs. (7) and (8)). Then (22) becomes

$$\frac{d}{dt}\left\{\int\frac{\mathbf{E}_{T2}^2+\mathbf{B}_{T2}^2}{8\pi}dV\right\}=0 \quad (23)$$

Thus, for the closed system consisting of the electromagnetic field of a *nonstandard polarity*, the quantity with dimensions of an energy in brackets in this equation is conserved. We can therefore call the quantity $\left(\mathbf{E}_{T2}^2+\mathbf{B}_{T2}^2\right)/(8\pi)$ the energy density of the electromagnetic field (16).

Now, if we integrate over any *finite* volume, then the surface integral in (22) generally does not vanish, so that we can write the equation in the form

$$\frac{d}{dt}\left\{\int\frac{\mathbf{E}_{T2}^2+\mathbf{B}_{T2}^2}{8\pi}dV\right\}=-\oint\mathbf{S}_{T2}\cdot d\mathbf{f} \quad (24)$$

On the left stands the change in the total energy of field per unit time. Therefore the integral $\oint\mathbf{S}_{T2}\cdot d\mathbf{f}$ must be interpreted as the flux of field energy across the surface bounding the given volume, so that the vector $\mathbf{S}_{T2}$ is a density of this flux (the Poynting vector) the amount of field energy passing through unit area of the surface in unit time.

Thus we can see that the expressions for energy density and the Poynting vector were obtained without subjecting the field to any *polarity* conditions.

We are not going to consider in detail the T_2-solution of FME, instead, we emphasize that the waves formed by this solution spread in the opposite direction with respect the waves formed by the T_1-solution. Absolute magnitudes of the Poynting vectors for T_1 and T_2 coincide, while the distribution of the energy density is different.

2.3. Case NT: Non-transferring Energy Wave Solutions

Consider the case when in Eqs. (5) $C_1 = \pm C_2 = 1$. In this case the solutions (5) become

$$\mathbf{E}_{NT}=[\mathbf{e}(\mathbf{r})\pm\mathbf{b}(\mathbf{r})]\sin(\Omega t) \text{ and } \mathbf{B}_{NT}=[\pm\mathbf{e}(\mathbf{r})+\mathbf{b}(\mathbf{r})]\cos(\Omega t) \quad (25)$$

Here we just adduce the obvious list of unusual properties of this solution:

1) The vectors $\mathbf{E}_{NT}$ and $\mathbf{B}_{NT}$ do not have *any polarity*, in other words these electric and magnetic vectors are *neither* polar *nor* axial ones.

2) The vectors $\mathbf{E}_{NT}$ and $\mathbf{B}_{NT}$ are mutually collinear (this directly follows from Eq.(25)), and, consequently, the Poynting vector, corresponding to these electromagnetic fields, is zero in every point of space.

3. Recall that in this case $\frac{d}{dt}=\frac{\partial}{\partial t}$.

3) In spite of the fact that the electric and magnetic intensities depend on time, the energy density distribution is a constant with respect to time and depends on spatial coordinates only:

$$\mathrm{w}_{NT} = \frac{1}{8\pi}\left[|\mathbf{e}(\mathbf{r})|^2 + |\mathbf{b}(\mathbf{r})|^2\right]. \qquad (26)$$

For $r \to 0$ and $r \to \infty$ the energy density correspondingly becomes:

$$\lim_{r\to 0} \mathrm{w}_{NT} = \frac{D^2\Omega^6}{18\pi c^6} \text{ and } \lim_{r\to\infty} \mathrm{w}_{NT} = 0 \qquad (27)$$

However, the integral of w_{NT} over *all* space diverges. It means that all energy of the electromagnetic fields (25) in infinite space is infinite.

Thus, we obtained from FME the unusual free electromagnetic fields which oscillate in every point of space but are not traveling waves and, correspondingly, do not transfer energy and momentum (that is why we can call them "standing" waves).

One of these solutions ($C_1 = C_2 = 1$ corresponds to the solution of FME found by Rodrigues and Maiorino (1996) and Chu and Ohkawa (1982):

$$\mathbf{E}_{NT} = \mathbf{a}(\mathbf{r})\sin(\Omega t) \text{ and } \mathbf{B}_{NT} = \mathbf{a}(\mathbf{r})\cos(\Omega t) \qquad (28)$$

where vector a(r) satisfies the vector equation rot $\mathbf{a}(x, y, z) = \frac{\Omega}{c}\mathbf{a}(x, y, z)$.

These fields are formed by vectors having no polarity, the energy density of these fields does not change with time, waves of this field do not transfer energy and momentum. But these electromagnetic fields *are a solution of the free Maxwell equation*!

2. Non-equivalence of the Poynting Vector and the Energy Flux (Umov) Vector in Obtained Wave Solutions

More often than not students of physics suppose that the Poynting vector

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} \qquad (29)$$

and Umov (energy flux) vector[4]

$$\mathbf{U} = \mathrm{w}c\mathbf{n} \qquad (30)$$

always coincide for electromagnetic wave spreading in vacuum in every point. Here **n** is a unit vector along the direction of propagation of the wave, c is the velocity of light (i.e. the transferring energy velocity) and w is the energy density of the electromagnetic wave. In actual fact, this assertion is proved at least for plane and spherical electromagnetic waves in vacuum.[5] Nevertheless, the assertion that **S** = **U** for waves of a more general kind is not proved in textbooks and monographs.

Let us study what condition in vacuum for **E** and **B** in an electromagnetic wave must be performed when the equality **S** = **U** is valid. We have in CGS (Gauss' system):

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} = \frac{c}{4\pi}EB\sin\alpha\,\mathbf{n} \qquad (31)$$

and

$$\mathbf{U} = \mathrm{w}c\mathbf{n} = \frac{c}{8\pi}\left(E^2 + B^2\right)\mathbf{n}. \qquad (32)$$

Equating (31) and (32) we obtain that

$$2EB\sin\alpha = E^2 + B^2 \qquad (33)$$

or

$$(E - B)^2 + 2EB(1 - \sin\alpha) = 0 \qquad (34)$$

where α is the angle between **E** and **B**. Last equality (34) can be valid if and only if $E = B$ and $\alpha = \pi/2$. Thus for the equality of the Poynting vector and Umov vector it is necessarily and sufficiently that $\mathbf{E}\perp\mathbf{B}$ and $E = B$.

If we hark back to the solutions (12) and (13) of the FME then one can make certain that these fields do not satisfy the condition $E = B$. Theoretically it can mean that in this kind of waves (which, obviously, are neither spherical nor plane) either the Poynting vector or the Umov vector (or both) cannot describe the energy flux density of the given waves.

As another example let us consider a plane polarized electromagnetic wave spreading across the area where a constant homogeneous electric field $\mathbf{E}_c$ is present. Let the constant field $\mathbf{E}_c$ be collinear to the variable electric field of the wave ($\mathbf{E}_w$) in every point at any instant of time.

4. The Umov vector (**U** = w**v**) is a more general vector (unlike the Poynting vector) describing the energy flux density of *any* kind of energy, w is the density of a corresponding kind of the energy (not excepting electromagnetic energy) and **v** is a velocity of spreading of the energy in a given point. Thus the Poynting vector formally must be a particular case of the Umov vector.

5. See, *e.g.* classical textbooks. Tamm (1975) or Landau and Lifshitz (1973).

Without taking into account the existence of the field $\mathbf{E}_c$, in this area absolute values of electric and magnetic components of the wave are equal ($E_w = B_w$) and[6]

$$\mathbf{B}_w = \mathbf{n}\times\mathbf{E}_w. \tag{35}$$

However after "turning on" the field $\mathbf{E}_c$, the absolute value of the resultant electric component $\mathbf{E}_r$ of the wave changes:

$$\mathbf{E}_r = \mathbf{E}_w + \mathbf{E}_c \tag{36}$$

where $\mathbf{E}_w$ is the electric component of the wave without $\mathbf{E}_c$. It is obvious that the resultant magnetic component $\mathbf{B}_r$ does not change when one takes into account the field $\mathbf{E}_c$, i.e.

$$\mathbf{B}_p + \mathbf{B}_w. \tag{37}$$

Let us now calculate the Poynting and Umov vectors (31), (32) (for $\alpha = \pi/2$) taking into account that according to the superposition principle fields **E** and **B** must be *resultant* fields:

$$\mathbf{S} = \frac{c}{4\pi}(\mathbf{E}_r \times \mathbf{B}_w). \tag{38}$$

Thus taking into account Eqs. (34)-(36) we obtain:

$$\mathbf{S} = c\frac{E_w^2 + B_w^2}{8\pi}\mathbf{n} + c\frac{2E_\chi E_w}{8\pi}\mathbf{n}, \tag{39}$$

and in turn

$$\mathbf{U} = c\frac{E_w^2 + B_w^2}{8\pi}\mathbf{n} + c\frac{2E_\chi E_w}{8\pi}\mathbf{n} + c\frac{2E_c^2}{8\pi}\mathbf{n} \tag{40}$$

Of course, the integration of the expressions (39) and (40) along of some closed surfaces gives the equal result.

Thus the following questions appear:

1. Why do the Poynting and Umov vectors, while they are defined both as a density of a flux of energy, not coincide?

2. Must fields (like $\mathbf{E}_c$, for example), which have no relation to the electromagnetic wave, be a component part of the expression for the energy-flux vectors (39), (40)?[7]

3. How does the ambiguousness of the Poynting vector affect calculation of the energy flux across *open* surfaces?

According to the superposition principle, in order to calculate fields in a given point one has to take into account all fields (i.e. both wave fields and non-wave ones) but it leads to inconsistent results. Many authors (see, e.g., Landau and Lifshitz, 1973: 10 x 47: 111) just exclude non-wave fields from consideration while it is not completely right from the superposition principle point of view.

Thus one can reach a conclusion about the existence of a disagreement between the superposition principle and energetic characteristics of electromagnetic fields.

3. Simple Solving of the System (6)

In order to solve this system, let us first note that formally summing two equations (6) we obtain

$$\nabla\times(\mathbf{e}+\mathbf{b}) = \frac{\Omega}{c}(\mathbf{e}+\mathbf{b}) \text{ or } \nabla\times\mathbf{a} = \frac{\Omega}{c}\mathbf{a}\,. \tag{41}$$

So, at first we resolve Eq. (41) with respect to **a**, and then we obtain from the vector **a** (which, obviously, has no polarity) the polar vector **e** and the axial vector **b**. Actually, one can express polar and axial parts of any vector without polarity, in general, as follows:

$$\mathbf{e}(\mathbf{r}) = \frac{1}{2}[\mathbf{a}(\mathbf{r}) - \mathbf{a}(-\mathbf{r})] \tag{42}$$

and

$$\mathbf{b}(\mathbf{r}) = \frac{1}{2}[\mathbf{a}(\mathbf{r}) + \mathbf{a}(-\mathbf{r})] \tag{43}$$

Now, if we calculate a rotor of both parts of equations (42), (43) one can be satisfied that the system (6) is fulfilled:

$$\nabla\times\mathbf{e}(\mathbf{r}) = \frac{1}{2}[\nabla\times\mathbf{a}(\mathbf{r}) - \nabla\times\mathbf{a}(-\mathbf{r})] = \frac{1}{2}[\frac{\Omega}{c}\mathbf{a}(\mathbf{r}) + \frac{\Omega}{c}\mathbf{a}(-\mathbf{r})] = \frac{\Omega}{c}\mathbf{b}(\mathbf{r}) \tag{44}$$

and

$$\nabla\times\mathbf{b}(\mathbf{r}) = \frac{1}{2}[\nabla\times\mathbf{a}(\mathbf{r}) + \nabla\times\mathbf{a}(-\mathbf{r})] = \frac{1}{2}[\frac{\Omega}{c}\mathbf{a}(\mathbf{r}) - \frac{\Omega}{c}\mathbf{a}(-\mathbf{r})] = \frac{\Omega}{c}\mathbf{e}(\mathbf{r}). \tag{45}$$

Here we take into account that after inverting the coordinates, the equation $\nabla\times\mathbf{a}(\mathbf{r}) = \frac{\Omega}{c}\mathbf{a}(-\mathbf{r})$ becomes $-\nabla\times\mathbf{a}(-\mathbf{r}) = \frac{\Omega}{c}\mathbf{a}(-\mathbf{r})$. Thus, one can see that if we find **a** as a solution of Eq. (41) it means that we find **e** and **b** as solution of the system (6).

In spite of the fact that equation (41) was already solved in the literature (Rodríguez and Maiorino, 1996; Chu and

6. See Landau and Lifshitz (1973), § 47. In our case $\varepsilon=\mu=1$.

7. See Landau and Lifshitz (1973), § 31. We just note that the expressions for the Poynting vector **S** and for the energy density w were obtained without taking into account the *origin* of the electric and magnetic fields in these expressions.

Ohkava, 1982) we decide to adduce here a different and very simple method of the solution of this vector equation.

One can satisfy oneself that a simple way to obtain a solution of Eq.(41) exists, if we represent the vector in the spherical system of coordinates as an axial-symmetrical vector:

$$\mathbf{a} = a_r(r,\theta)\mathbf{e}_r + a_\theta(r,\theta)\mathbf{e}_\theta + a_\varphi(r,\theta)\mathbf{e}_\varphi . \quad (46)$$

The rotor of a vector in spherical coordinates out of the origin is:

$$\nabla\times a = \frac{\mathbf{e}_r}{r^2\sin\theta}\left(\frac{\partial(ra_\varphi\sin\theta)}{\partial\theta} - \frac{\partial(ra_\varphi)}{\partial\varphi}\right) + \frac{\mathbf{e}_\theta}{r\sin\theta}\left(\frac{\partial(a_r)}{\partial\varphi} - \frac{\partial(ra_\varphi\sin\theta)}{\partial r}\right) \quad (47)$$

Taking into account Eq. (41) and comparing (46) and (47) we obtain the following system:

$$\left.\begin{aligned} \frac{\partial(a_\varphi\sin\theta)}{\partial\theta} &= \frac{\Omega r a_r\sin\theta}{c} \\ \frac{\partial(ra_\varphi)}{\partial r} &= -\frac{\Omega r a_\theta}{c} \\ \frac{\partial(ra_\theta)}{\partial r} - \frac{\partial a_r}{\partial\theta} &= \frac{\Omega r a_\varphi}{c} \end{aligned}\right\}. \quad (48)$$

From the system (48) one can obtain a differential equation for a_φ only:

$$r\frac{\partial^2}{\partial r^2}(ra_\varphi) + \frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(a_\varphi\sin\theta)\right) + \frac{\Omega^2 r^2}{c^2}a_\varphi = 0 \quad (49)$$

If we look for the solution of Eq. (49) in the form

$$a_\varphi = R(r)\Theta(\theta) \quad (50)$$

we obtain that these functions have to satisfy the following equations respectively:

$$r^2\frac{d^2(rR)}{dr^2} + \left(\frac{\Omega^2 r^2}{c^2} + p\right)rR = 0 \quad (51)$$

and

$$\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{d}{d\theta}(\Theta\sin\theta)\right) - p\Theta = 0 \quad (52)$$

where p is an arbitrary constant. If p were zero, the solution for rR in Eq. (51) would be $A\cos\frac{\Omega r}{c} + B\sin\frac{\Omega r}{c}$ (A and B are constants). Accordingly, in general, we are going to look for the solution of Eq.(51) in the form

$$rR = A(r)\cos\frac{\Omega r}{c} + B(r)\sin\frac{\Omega r}{c} \quad (53)$$

here $A(r)$ and $B(r)$ are some functions of r. Substituting (53) in Eq.(51) and taking into account that coefficients of sine and cosine (which have the same argument) must be equal to *zero* separately, we obtain a system of two ordinary differential equations:

$$A'' + \frac{p}{r^2}A + \frac{2\Omega}{c}B' = 0 \text{ and } B'' + \frac{p}{r^2} - B\frac{2\Omega}{c}A' = 0 . \quad (54)$$

Let us propose that $A(r) = \mu r^m$ and $B(r) = \nu r^n$, where μ, ν, m, n are constants and m, n enter. Substituting these values in Eqs.(54) we obtain ''characteristic'' equations:

$$\mu m(m-1) + p\mu + \frac{2\Omega}{c}\nu n r^{n-m+1} = 0 \quad (55)$$

$$\nu n(n-1) + p\nu - \frac{2\Omega}{c}\mu m r^{m-n+1} = 0 ,$$

that one can verify only in the two following cases:
I. $m = 0$, $n = -1$, $p = -2$, $\mu = -\frac{\Omega}{c}\nu$, and taking into account Eq.(53) we obtain for $\nu = 1$

$$R = \frac{1}{r^2}\left(-\frac{\Omega r}{c}\cos\frac{\Omega r}{c} + \sin\frac{wr}{c}\right); \quad (56)$$

II. $m = -1$, $n = 0$, $p = -2$, $\nu = -\frac{\Omega}{c}\mu$, and taking into account Eq. (53) we obtain for $\mu = 1$

$$R = \frac{1}{r^2}\left(\cos\frac{\Omega r}{c} + \frac{\Omega r}{c} + \sin\frac{\Omega r}{c}\right). \quad (57)$$

So the general solution of Eq. (51) for Rr is:

$$Rr = \frac{C_1}{r^2}\left(-\frac{\Omega r}{c}\cos\frac{\Omega r}{c} + \sin\frac{\Omega r}{c}\right) + \frac{C_2}{r^2}\left(\cos\frac{\Omega r}{c} + \frac{\Omega r}{c}\sin\frac{\Omega r}{c}\right), \quad (58)$$

where C_1 and C_2 are arbitrary constants. The solution (58) one can express in the form

$$Rr = \frac{C}{r^2}\left\{\cos\left(\frac{\Omega r}{c} - \delta\right) + \frac{\Omega r}{c}\sin\left(\frac{\Omega r}{c} - \delta\right)\right\}, \quad (59)$$

where C and δ are arbitrary constants.

Now Eq. (52) becomes ($p = -2$):

$$\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{d}{d\theta}(\Theta\sin\theta)\right) + 2\Theta = 0 \quad (60)$$

Its general solution is:

$$\Theta(\theta) = c_3\sin\theta + c_4\left(\cot\theta + \sin\theta\ln\left|\csc\theta - \cot\theta\right|\right) \quad (61)$$

As a particular case we take the values $C_4 = 0$ (because a corresponding solution has a singularity in $\theta = (2n+1)\pi$)

and $C_3 = 1$ (by virtue of homogeneity of the equation for the vector **a**).

Thus, we can write the solution (50) as follows:

$$a_\varphi(r,\theta) = \frac{\alpha}{r^2}\sin\theta \tag{62}$$

where

$$\alpha = \cos\left(\frac{\Omega r}{c} - \delta\right) + \frac{\Omega r}{c}\sin\left(\frac{\Omega r}{c} - \delta\right)$$

Now, using Eqs. (48), we can find $a_r(r, \theta)$ and $a_\theta(r, \theta)$:

$$a_r(r,\theta) = \frac{2c\alpha}{\Omega r^3}\cos\theta, \quad a_\theta(r,\theta) = \frac{c\gamma}{\Omega r^3}\sin\theta \tag{63}$$

where

$$\gamma = \alpha - \frac{\Omega^2 r^2}{c^2}\cos\left(\frac{\Omega r}{c} - \delta\right).$$

And so, we have found the solution of Eq. (41) which in the spherical system of coordinates is:

$$\mathbf{a} = D\left\{\frac{2\alpha}{r^3}\cos\theta\right\}\mathbf{e}_r + D\left\{\frac{\gamma}{r^3}\sin\theta\right\}\mathbf{e}_\theta + D\left\{\frac{\Omega\alpha}{cr^2}\sin\theta\right\}\mathbf{e}_\varphi \tag{64}$$

where for convenience we multiplied the solution by $D\frac{\Omega}{c}$. Then using (64), (42) and (43) we obtain the solutions (7), (8) of the system (6).

Conclusion

There is a widely held view that, in Maxwell's Classical Electrodynamic Theory, all has been done, all is understood and that nothing further can be done other than carry out calculations for ever more complex systems, that is why it is necessary to move towards more fundamental physical theories. This seems to be confirmed as quantum physics was created out of the basis of this theory and thus, quantum electrodynamics was born. These two branches of physics use classical electrodynamics as an impeccable instrument. Why concern ourselves with the basis of classical electrodynamics when quantum mechanics and quantum electrodynamics, with all their achievements, have already demonstrated their right to their own existence?

However, upon thoroughly analyzing the basics of classical electrodynamics, almost at every step along the way we have a sense of internal dissatisfaction. This feeling comes from the conceptual problems which remain unresolved by this theory, and which we have become accustomed to avoiding in silence.[8] There are various problems of this type[9] but we limited ourselves to those related to this work: the incompatibility of the definition of electromagnetic energy density and the principle of the superposition of the electromagnetic fields; the lack of uniqueness in the definition of the vector for the density of energy flow along open surfaces; the lack of coincidence between the Umov vector and the Poynting vector; the transversal character of electromagnetic waves, demonstrated for a certain kind of electromagnetic waves only; the possibility of a nonstandard polarity of the solutions of the *free* Maxwell equations in vacuum etc.

In this article, on the basis of the solutions described, we wished to show that not all is understood in the theory of classical electrodynamics. This situation allows us the possibility of finding unexpected elements, which can fuel the imagination and analysis. Generally, in texts relating to electrodynamics examples confirming the "well-established" facts of the theory are described. However, we consider that from a pedagogical viewpoint, the counter-examples, which show that not all is well established, are also necessary for a deeper understanding of the fundamentals of this far from finished theory.

In this manner, on the basis of the solutions found (5) for the equations of the electromagnetic field in a vacuum, we show that certain "standard properties" generally imposed on these solutions are not always fulfilled. The lack of the polarity condition of these solutions at microscopic level could indicate that the Maxwell equations in a vacuum may not necessarily be a consequence of these equations in medium with charges and currents. The latter, as shown here, can lead to wave solutions which do not

8. While R. Feynman (1964) wrote "[...] this tremendous edifice (classical electrodynamics) which is such a beautiful success in explaining so many phenomena, ultimately falls on its face. When you follow any of our physics too far, you find that it always gets into some kind of trouble. [...] the failure of the classical electromagnetic theory. [...]Classical mechanics is a mathematically consistent theory; it just doesn't agree with experience. It is interesting, though, that the classical theory of electromagnetism is an unsatisfactory theory all by itself. There are difficulties associated with the ideas of Maxwell's theory which are not solved by and not directly associated with quantum mechanics [...]".

9. See, for example, a purposeful review of this kind of works "Essay on Non-Maxwellian Theories of Electromagnetism" by V. V. Dvoeglazov, Hadronic J. Suppl. , 241 (1997). See also brilliant monographs: O.D. Jefimenko, "Causality, Electromagnetic Induction, and Gravity" (Electret Scientific, Star City,1992); O.D. Jefimenko, "Electricity and Magnetism", 2nd ed.(Electret Scientific, Star City,1989); O.D. Jefimenko, "Electromagnetic retardation and Theory of Relativity" (Electret Scvientific, Star City, 1997).

transmit energy, as well as solutions where the electrical field is parallel to the magnetic field.

Even if we restrict ourselves to the analysis of solutions with "correct" polarity, the Maxwell equations lead to unexpected results. It is surprising that the solution (9) has strange properties such as the existence of longitudinal stationary magnetic waves along the -axis. And if this is not surprising, let us look, in these solutions, at the existence of immobile spheres with constant form and dimensions, in which the total electromagnetic energy is constant. These circumstances make reference to the little studied processes of third-dimensional interference. Furthermore, the existence of magnetic spheres of this kind opens up the possibility of the solution of "practical" problems such as the confinement of plasma, the explanation of the enigmatic phenomenon known as ball-lightning (for example one can consider these solutions as a certain mathematical rationale of the Kapitsa's hypothesis about interference nature of this phenomenon Kapitsa, 1955) and also, who knows? the solution to some theoretical problems such as the construction of a classical model of the atom, and even a correction of the model of the internal structure of the stars.

On the other hand, concepts such as the density of energy flux and the density of energy for the electromagnetic field are still incoherent. By applying these concepts to the fields, we come into contradiction with the general principle of superposition. Furthermore, the natural definition of the energy flux with the help of the Umov vector and the definition of this same concept from the law of energy conservation (that is, the Poynting vector), in a curious manner only coincide under certain restrictions on the superposed electromagnetic fields. Theoretically both definitions of the energy flux vectors, for the Poynting vector as well as for the Umov, are not unique: for the Poynting vector because of the possibility of adding a rotor of an arbitrary vector and for the Umov because of ambiguousness of the definitions of its direction, which, obviously, must coincide with the direction of the Poynting vector.

Here, as in all cases, experiment has the final word.[11] Finally, it is significant that all unusual properties of wave solutions of free Maxwell equations discussed in the present work take place exclusively in the reference frame which is at rest relative to ''magnetic spheres'' described above.

10. Nevertheless J.D.Jackson in his well-known textbook (Jackson, 1998: 259), says ''The vector **S**, representing energy flow, is called the Poynting vector. It is given by **S = E** x **H** (6.109) [...] Since only its divergence appears in the conservation law, the Poynting vector seems arbitrary to the extent that the curl of any vector field can be added to it. Such an added term can, however, have no physical consequences. Relativistic considerations (Section 12.10) show that (6.109) is unique''.

11. See, for example, the recent work by Chubykalo, Espinoza and Tzonchev (2004).

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