On Pillai's problem with the Fibonacci and Pell sequences

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§1. Introduction. Let $\mathbf{U} := (U_n)_{n\geqslant 0}$ and $\mathbf{V} := (V_n)_{n\geqslant 0}$ be two linearly recurrent sequences of integers. Recently, the following variation of a problem of Pillai has been studied. Find all non-negative integer solutions (n, m, n_1, m_1) of the equation

$$U_n - V_m = U_{n_1} - V_{m_1}, \qquad (n, m) \neq (n_1, m_1). \tag{1}$$

In particular, find also all integers c which can be written as the difference between an element of $\mathbf{U}$ and an element of $\mathbf{V}$ in at least two different ways. Pillai [12], studied this problem when $\mathbf{U}$ and $\mathbf{V}$ are the sequences of powers of a, and powers of b, respectively, where a, b are two given coprime integers different than 0, ± 1. It has been shown in [4] that, under some technical but natural conditions, equation (**1**) has only finitely many non-negative integer solutions and all of them are effectively computable. This version of Pillai's problem was initiated in [7] by Ddamulira, Luca and Rakotomalala who studied equation (**1**) when $\mathbf{U}$ and $\mathbf{V}$ are the sequences of Fibonacci numbers and powers of 2, respectively. Many other particular cases have been studied. See, for example [3], [6], [8]. We recall that the Fibonacci sequence $(F_n)_{n\geqslant 0}$ is given by $F_0 = 0$, $F_1 = 1$ and the recurrence formula

$$F_{n+2} = F_{n+1} + F_n \qquad \text{for all} \qquad n \geq 0.$$

Let $(P_n)_{n\geqslant 0}$ be the Pell sequence given by $P_0 = 0$, $P_1 = 1$, and the recurrence formula

$$P_{n+2} = 2P_{n+1} + P_n \qquad \text{for all} \qquad n \geq 0.$$

Their first terms are,

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, \ldots$$

and

$$0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, \ldots,$$

respectively. In this note, we study another particular case of this problem, namely equation (**1**) with Fibonacci and Pell numbers. More precisely, we look at the equation

$$F_n - P_m = F_{n_1} - P_{m_1} \tag{2}$$

in integer pairs $(n, m) \neq (n_1, m_1)$. Since $F_1 = F_2 = 1$, we assume that $n \neq 1,\ n_1 \neq 1$. That is, whenever we think of 1 as a member of the Fibonacci sequence, we think of it as being F_2. Our result is then the following

Theorem 1. *All solutions non-negative integer solutions* (n, m, n_1, m_1) *of* (**2**) *with* $n \neq 1,\ n_1 \neq 1$ *belong to the set*

$$\left\{\begin{array}{llll} (2,1,0,0), & (2,2,0,1), & (3,1,2,0), & (3,2,0,0), \\ (3,2,2,1), & (4,1,3,0), & (4,2,2,0), & (4,2,3,1), \\ (4,3,0,2), & (5,2,4,0), & (5,3,0,0), & (5,3,2,1), \\ (5,3,3,2), & (6,3,4,0), & (6,3,5,2), & (6,4,2,3), \\ (7,3,6,0), & (7,4,2,0), & (7,4,3,1), & (7,4,4,2), \\ (9,5,5,0), & (11,6,8,2), & (16,9,3,0), & (16,9,4,1) \end{array}\right\}.$$

The set of integers c *admitting two representations as a difference between a Fibonacci and a Pell number in at least two different ways is*

$$\{-4,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 5,\ 8,\ 19\}.$$

The representations of the above c *are*

$$\begin{array}{rcl} -4 & = & F_6 - P_4 = F_2 - P_3; \\ -2 & = & F_4 - P_3 = F_0 - P_2; \\ -1 & = & F_2 - P_2 = F_0 - P_1; \\ 0 & = & F_5 - P_3 = F_3 - P_2 = F_2 - P_1 = F_0 - P_0; \\ 1 & = & F_7 - P_4 = F_4 - P_2 = F_3 - P_1 = F_2 - P_0; \\ 2 & = & F_{16} - P_9 = F_4 - P_1 = F_3 - P_0; \\ 3 & = & F_6 - P_3 = F_5 - P_2 = F_4 - P_0; \\ 5 & = & F_9 - P_5 = F_5 - P_0; \\ 8 & = & F_7 - P_3 = F_6 - P_0; \\ 19 & = & F_{11} - P_6 = F_8 - P_2. \end{array}$$

§2. Tools. The first one is a lower bound for a linear forms in logarithms due to Matveev [11]. Let α be an algebraic number of degree d. Let a be the leading coefficient of its minimal polynomial over $\mathbb{Z}$ and let $\alpha_1 = \alpha, \ldots, \alpha_d$ denote the conjugates of α. The Weil height of α is defined as

$$h(\alpha) = \frac{1}{d}\left(\log a + \sum_{i=1}^{d} \log \max\{|\alpha_i|, 1\}\right).$$

The height has the following basic properties. For α, β algebraic numbers and $m \in \mathbb{Z}$, we have:

- $h(\alpha + \beta) \leqslant h(\alpha) + h(\beta) + \log 2$.
- $h(\alpha\beta) \leqslant h(\alpha) + h(\beta)$.
- $h(\alpha^m) = |m| h(\alpha)$.

Now let $\mathbb{L}$ be a real number field of degree $d_{\mathbb{L}}$, $\alpha_1, \ldots, \alpha_\ell \in \mathbb{L}$ and $b_1, \ldots, b_\ell \in \mathbb{Z}\backslash\{0\}$. Let $B \geqslant \max\{|b_1|, \ldots, |b_\ell|\}$ and

$$\Lambda = \alpha^{b_1} \cdots \alpha^{b_\ell} - 1.$$

Let $A_1, \ldots, A_\ell$ be real numbers such that

$$A_i \geqslant \max\{d_{\mathbb{L}} h(\alpha_i), |\log \alpha_i|, 0.16\} \qquad \text{for all} \qquad i = 1, \ldots, \ell.$$

The following result is due to Matveev in [11] (see also Theorem 9.4 in [2]).

Theorem 2. *Assume that $\Lambda \neq 0$. Then*

$$\log |\Lambda| > -1.4 \times 30^{\ell+3} \times \ell^{4.5} \times d_{\mathbb{L}}^2 (1 + \log d_{\mathbb{L}})(1 + \log B) A_1 \cdots A_\ell.$$

In this paper, we always use $\ell = 3$. Further, $\mathbb{L} = \mathbb{Q}[\sqrt{2}, \sqrt{5}]$ has degree $d_{\mathbb{L}} = 4$. Thus, once for all we fix the constant

$$C := 5.46696 \times 10^{12} > 1.4 \times 30^{3+3} \times 3^{4.5} \times 4^2 (1 + \log 4).$$

Matveev's bound gives us some large bounds on our parameters. In order to lower such bounds, we use a version of a reduction method of Baker-Davenport based on Lemma in [1]. We shall use the one given by Bravo, Gomez and Luca in [5]. For a real number x, we write

$$\|x\| = \min\{|x - n| \ : \ n \in \mathbb{Z}\}.$$

Lemma 3. *Let M be a positive integer. Let $\tau,\ \mu,\ A > 0,\ B > 1$ be given real numbers. Assume that p/q is a convergent of τ such that $q > 6M$ and $\varepsilon := \|q\mu\| - M\|q\tau\| > 0$. Then the inequality*

$$0 < |n\tau - m + \mu| < \frac{A}{B^w}$$

does not have a solution in positive integers $n,\ m$ and w in the ranges

$$n \leqslant M \qquad \text{and} \qquad w \geqslant \frac{\log(Aq/\varepsilon)}{\log B}.$$

This lemma is a slightly variation of the one given by Dujella and Petho in [9]. The following lemma is also useful. It is Lemma 7 in [10].

Lemma 4. *If* $m \geqslant 1$, $T > (4m^2)^m$ *and* $T > x/(\log x)^m$, *then*

$$x < 2^m T(\log T)^m.$$

§3. Proof of Theorem 1. We start with some basic properties of our sequences. Put

$$\alpha := \frac{1+\sqrt{5}}{2}, \quad \beta := \frac{1-\sqrt{5}}{2}\ ; \qquad \text{and} \qquad \gamma := 1+\sqrt{2}, \quad \delta := 1-\sqrt{2}.$$

We have the well–known Binet's formulas

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \quad \text{and} \quad P_n = \frac{\gamma^n - \delta^n}{2\sqrt{2}} \tag{3}$$

which hold for all $n \geqslant 0$. Further, the inequalities

$$\alpha^{n-2} \leqslant F_n \leqslant \alpha^{n-1} \quad \text{and} \quad \gamma^{n-2} \leqslant P_n \leqslant \gamma^{n-1} \tag{4}$$

also hold for all $n \geqslant 1$.

Now, we study our equation (**2**) in non-negative integers (n, m, n_1, m_1) with $(n, m) \neq (n_1, m_1)$. As we said, we assume $n \neq 1$, $n_1 \neq 1$. It could happen that $\min\{n, n_1\} = 0$. At any rate, $\max\{n, n_1\} \geqslant 2$. If in (**2**) we have $m = m_1$, then $F_n = F_{n_1}$, implies that $n = n_1$, a contradiction. Thus, from now on we assume $m > m_1$. Rewriting (**2**) as

$$F_n - F_{n_1} = P_m - P_{m_1}, \tag{5}$$

we observe the right-hand side is positive. Hence, so is the left-hand side, therefore $n > n_1$. We now compare the two sides of (**5**) using (**4**). We have

$$\alpha^{n-4} \leqslant F_n - F_{n_1} = P_m - P_{m_1} \leqslant P_m \leqslant \gamma^{m-1}.$$

The left–hand side inequality is clear if $n_1 = 0$. It is also clear if $n_1 \neq 0$, since in that case $n_1 \geq 2$, so $n \geq 3$, so $F_n - F_{n_1} \geq F_n - F_{n-1} = F_{n-2} \geq \alpha^{n-4}$. Thus, $\alpha^{n-4} \leqslant \gamma^{m-1}$. In a similar way,

$$\alpha^{n-1} \geqslant F_n \geqslant F_n - F_{n_1} = P_m - P_{m_1} \geqslant P_{m-1} \geqslant \gamma^{m-3},$$

where the right–most inequality is clear (both for $m_1 = 0$ and for $m_1 > 0$). We thus have

$$n - 4 \leqslant (m-1)\frac{\log\gamma}{\log\alpha} \qquad \text{and} \qquad n - 1 \geqslant \frac{\log\gamma}{\log\alpha}(m-3). \tag{6}$$

Since $\log\gamma/\log\alpha = 1.8315709239\ldots$ it follows that if $n \leqslant 300$, then $m \leqslant 167$. Running a Mathematica program in the range $0 \leqslant n_1 < n \leqslant 300$ and $0 \leqslant m_1 < m \leqslant 167$, with our convention, we obtain all the possibilities listed in Theorem 1.

From now on, $n > 300$. Further, by (**6**) we get $m > 163$ and also $n > m$. From Binet's formulas (**3**), we obtain

$$\begin{aligned}\left|\frac{\alpha^n}{\sqrt{5}} - \frac{\gamma^m}{2\sqrt{2}}\right| &= \left|\frac{\alpha^{n_1}+\beta^n-\beta^{n_1}}{\sqrt{5}} - \frac{\gamma^{m_1}-\delta^{m_1}+\delta^m}{2\sqrt{2}}\right| \leqslant \frac{\alpha^{n_1}+2}{\sqrt{5}} + \frac{\gamma^{m_1}+2}{2\sqrt{2}} \\ &\leqslant 2\max\{\alpha^{n_1+2}, \gamma^{m_1+1}\}. \qquad (\mathbf{7})\end{aligned}$$

Dividing through by $\gamma^m/2\sqrt{2}$ we get

$$\left|\frac{4}{\sqrt{10}}\gamma^{-m}\alpha^n - 1\right| \leqslant \max\{\alpha^{n_1-n+9}, \gamma^{m_1-m+1}\}, \qquad (\mathbf{8})$$

where we have used that $\alpha^{n-4} \leqslant \gamma^{m-1}$ as well as the fact that $4\sqrt{2} < \lambda^2 < \alpha^4$. Let Λ be the expression inside the absolute value in in the left–hand side above. Observe that Λ is not zero. Indeed, otherwise $8/5 = \gamma^{2m}/\alpha^{2n}$ is both a unit (an algebraic integer whose reciprocal is also an algebraic integer) and a rational number, which is false since the only rational units are ± 1.
Now we apply Matveev's inequality with

$$\alpha_1 = \frac{4}{\sqrt{10}}, \quad \alpha_2 = \gamma, \quad \alpha_3 = \alpha, \quad b_1 = 1, \quad b_2 = -m, \quad b_3 = n.$$

We have $B = n$. Further, we have $h(\alpha_1) = (\log 8)/2$, $h(\alpha_2) = (\log\gamma)/2$ and $h(\alpha_3) = \log\alpha/2$. Thus, we may take $A_1 := 4.2, A_2 := 1.8$ and $A_3 := 1$ we obtain that

$$\log|\Lambda| > -C(1+\log n) \times 4.2 \times 1.8.$$

Comparing with (**8**) we obtain

$$\min\{(n-n_1-9)\log\alpha, (m-m_1-1)\log\gamma\} \leqslant 4.13302 \times 10^{13}(1+\log n). \qquad (\mathbf{9})$$

We next study each of these two possibilities.

Case 1. $\min\{(n-n_1)\log\alpha, (m-m_1)\log\gamma\} = (n-n_1)\log\alpha$.

To this case, we rewrite our equation as follows:

$$\begin{aligned}\left|\left(\frac{\alpha^{n-n_1}-1}{\sqrt{5}}\right)\alpha^{n_1} - \frac{\gamma^m}{2\sqrt{2}}\right| &= \left|\frac{\beta^n-\beta^{n_1}}{\sqrt{5}} - \frac{\gamma^{m_1}-\delta^{m_1}+\delta^m}{2\sqrt{2}}\right| \\ &\leqslant \frac{2}{\sqrt{5}} + \frac{\gamma^{m_1}+2}{2\sqrt{2}} < \gamma^{m_1+2}.\end{aligned}$$

Thus,

$$\left|\left(\frac{4(\alpha^{n-n_1}-1)}{\sqrt{10}}\right)\alpha^{n_1}\gamma^{-m}-1\right|<\gamma^{m_1-m+4}. \qquad (\mathbf{10})$$

Let Λ_1 be the expression inside the absolute value which is in the left–hand side. We note that $\Lambda_1 \neq 0$, for if this is not so then we would get

$$\frac{\alpha^n-\alpha^{n_1}}{\gamma^m}=\frac{\sqrt{10}}{4},$$

which implies that the right–hand side is an algebraic integer, which it isn't (it's square is 5/8). We apply again Matveev's inequality by taking

$$\alpha_1=\frac{4(\alpha^{n-n_1}-1)}{\sqrt{10}}, \quad \alpha_2=\gamma, \quad \alpha_3=\alpha, \quad b_1=1, \quad b_2=-m, \quad b_3=n_1.$$

Thus, $B=n$. The heights of α_2 and α_3 have already been calculated. As for $h(\alpha_1)$, we have

$$\begin{aligned} h\left(\frac{4(\alpha^{n-n_1}-1)}{\sqrt{10}}\right) &\leqslant h\left(\frac{4}{\sqrt{10}}\right)+h\left(\alpha^{n-n_1}-1\right)\leqslant \frac{\log 8}{2}+h(\alpha^{n-n_1})+\log 2\\ &= \frac{\log 32}{2}+(n-n_1)\frac{\log\alpha}{2}\leqslant \frac{4.13304\times 10^{13}(1+\log n)}{2}, \end{aligned}$$

where we have used (**9**). Thus, we can take $A_1 := 8.26608\times 10^{13}(1+\log n)$, A_2 and A_3 as in the analysis of Λ, and get

$$\log|\Lambda_1| > -C\times(8.26608\times 10^{13}(1+\log n)^2)\times 1.8.$$

Combining this with (**10**), we get

$$(m-m_1)\log\gamma < 8.13424\times 10^{26}(1+\log n)^2.$$

Case 2. $\min\{(n-n_1)\log\alpha,(m-m_1)\log\gamma\}=(m-m_1)\log\gamma$.

Here, we rewrite our equation as

$$\begin{aligned} \left|\frac{\alpha^n}{\sqrt{5}}-\left(\frac{\gamma^{m-m_1}-1}{2\sqrt{2}}\right)\gamma^{m_1}\right| &= \left|\frac{\beta^n+\alpha^{n_1}-\beta^{n_1}}{\sqrt{5}}-\frac{\delta^m-\delta^{m_1}}{2\sqrt{2}}\right|\\ &\leqslant \frac{\alpha^{n_1}+2}{\sqrt{5}}+\frac{1}{\sqrt{2}}<\alpha^{n_1+5}. \end{aligned}$$

Thus,

$$\left|1-\left(\frac{\sqrt{10}(\gamma^{m-m_1}-1)}{4}\right)\gamma^{m_1}\alpha^{-n}\right|<\alpha^{n_1-n+7}. \qquad (\mathbf{11})$$

We let Λ_2 be the expression inside the absolute value in the left-hand side. As before, $\Lambda_2 \neq 0$, for otherwise we get that $8/5$ is an algebraic integer, which is false. We apply again Matveev's inequality by taking

$$\alpha_1 = \frac{\sqrt{10}(\gamma^{m-m_1}-1)}{4}, \quad \alpha_2 = \gamma, \quad \alpha_3 = \alpha, \quad b_1 = 1, \quad b_2 = m_1, \quad b_3 = -n.$$

Thus, $B = n$. The heights of α_2 and α_3 have already been calculated. As for $h(\alpha_1)$, we have

$$\begin{aligned} h\left(\frac{\sqrt{10}(\gamma^{m-m_1}-1)}{4}\right) &\leqslant h\left(\frac{\sqrt{10}}{4}\right) + h\left(\gamma^{m-m_1}-1\right) \\ &\leqslant \frac{4.13304 \times 10^{13}(1+\log n)}{2}, \end{aligned}$$

Thus, we can take the same A_1 as in Case 1, and so we get the same lower bound for $\log|\Lambda_2|$. Therefore,

$$(n-n_1)\log\gamma < 8.13424 \times 10^{26}(1+\log n)^2.$$

So, we have proved that

$$\max\{(n-n_1)\log\alpha, (m-m_1)\log\gamma\} \leqslant 8.13424 \times 10^{26}(1+\log n)^2. \qquad (\mathbf{12})$$

We now get a bound on n. Using Binet's formulas (**3**), we write our equation as follows:

$$\left|\frac{\alpha^{n-n_1}-1}{\sqrt{5}}\alpha^{n_1} - \frac{\gamma^{m-m_1}-1}{2\sqrt{2}}\gamma^{m_1}\right| = \left|\frac{\beta^n-\beta^{n_1}}{\sqrt{5}} - \frac{\delta^m-\delta^{m_1}}{2\sqrt{2}}\right| < \frac{2}{\sqrt{5}} + \frac{1}{\sqrt{2}} < 2.$$

Dividing across by $(\gamma^m - \gamma^{m-1})/2\sqrt{2}$, we obtain

$$\left|\left(\frac{4}{\sqrt{10}}\left(\frac{\alpha^{n-n_1}-1}{\gamma^{m-m_1}-1}\right)\right)\gamma^{-m_1}\alpha^{n_1} - 1\right| < \frac{4\sqrt{2}}{\gamma^m - \gamma^{m_1}} < \frac{8\sqrt{2}}{\gamma^m} < \frac{1}{\alpha^{n-8}}, \qquad (\mathbf{13})$$

where we used $\alpha^{n-4} < \gamma^{m-1}$, as well as the fact that $8\sqrt{2} < \alpha^4\gamma$. We let Λ_3 be the expression inside the absolute value in (**13**). We apply Matveev's inequality with

$$\alpha_1 = \frac{4}{\sqrt{10}}\left(\frac{\alpha^{n-n_1}-1}{\gamma^{m-m_1}-1}\right), \ \alpha_2 = \gamma, \ \alpha_3 = \alpha, \ b_1 = 1, \ b_2 = -m_1, \ b_3 = -n_1.$$

Thus, we take $B = n$. We need to show that $\Lambda_3 \neq 0$. To do this we take the $\mathbb{Q}$-automorphism σ of $\mathbb{L}$ given by $\sigma(\sqrt{5}) = -\sqrt{5}$ and $\sigma(\sqrt{2}) = \sqrt{2}$. Under this automorphism, we have $\sigma(\alpha) = \beta$, $\sigma(\gamma) = \gamma$ and $\sigma(\sqrt{10}) = -\sqrt{10}$. Thus, if $\Lambda_3 = 0$, then $\sigma(L_3) = 0$, which implies, in particular, that

$$\frac{\sqrt{10}}{4} = \left|\frac{\beta^n - \beta^{n_1}}{\gamma^m - \gamma^{m_1}}\right| < \frac{2}{\gamma^m(\gamma-1)} < \frac{1}{2},$$

since $m > 163$, which is a contradiction. As before, the heights of α_2 and α_3 have already been calculated. For $h(\alpha_1)$, we have

$$\begin{aligned} h\left(\frac{4}{\sqrt{10}}\left(\frac{\alpha^{n-n_1}-1}{\gamma^{m-m_1}-1}\right)\right) &\leqslant h\left(\frac{4}{\sqrt{10}}\right) + h\left(\alpha^{n-n_1}+1\right) + h\left(\gamma^{m-m_1}+1\right) \\ &\leqslant \frac{\log 128}{2} + (n-n_1)\frac{\log\alpha}{2} + (m-m_1)\frac{\log\gamma}{2} \\ &\leqslant 8.13425 \times 10^{26}(1+\log n)^2. \end{aligned}$$

Thus, we can take $A_1 := 3.25368 \times 10^{27}(1+\log n)^2$, and A_2, A_3 as before. Therefore, we get

$$\begin{aligned} \log|\Lambda_3| &> -C(1+\log n) \times (3.25368 \times 10^{27}(1+\log n)^2) \times 1.8 \\ &> -3.20181 \times 10^{40}(1+\log n)^3, \end{aligned}$$

which, upon comparing it to (**13**) and applying Lemma 4, we obtain

$$n < 3.77669 \times 10^{48}. \qquad (\mathbf{14})$$

Now, we will reduce the upper bound of n. To do this, let Γ be defined as

$$\Gamma = n\log\alpha - m\log\gamma + \log\left(\frac{4}{\sqrt{10}}\right).$$

Assume first that $\min\{n-n_1, m-m_1\} \geqslant 20$. We note that $\Lambda = e^{\Gamma} - 1 \neq 0$, so $\Gamma \neq 0$. If $\Gamma > 0$ then

$$0 < \Gamma < e^{\Gamma} - 1 = \Lambda = |\Lambda| < \max\{\alpha^{n_1-n+9}, \gamma^{m_1-m+1}\}.$$

On the other hand, if $\Gamma < 0$, we then have $1 - e^{\Gamma} = |e^{\Gamma} - 1| < 1/2$ which implies $e^{|\Gamma|} < 2$. Thus,

$$0 < |\Gamma| < e^{|\Gamma|} - 1 = e^{|\Gamma|}|\Lambda| < 2\max\{\alpha^{n_1-n+9}, \gamma^{m_1-m+1}\}.$$

So, in both cases we have

$$0 < |\Gamma| < 2\max\{\alpha^{n_1-n+9}, \gamma^{m_1-m+1}\}. \qquad (\mathbf{15})$$

Dividing through by $\log\gamma$ in the above inequality, we get

$$0 < |n\tau - m + \mu| < \max\left\{\frac{175}{\alpha^{n-n_1}}, \frac{6}{\gamma^{m-m_1}}\right\},$$

where

$$\tau := \frac{\log \alpha}{\log \gamma}, \qquad \mu := \frac{\log\left(4/\sqrt{10}\right)}{\log \gamma}.$$

Now we apply Lemma 3. To do this, we take $M := 3.77669 \times 10^{48}$ (a bound on m and n by (**14**)) our τ and, with a *Mathematica* program, we find that the denominator of the convergent

$$\frac{p_{112}}{q_{112}} = \frac{111842821415068814601069451383096958405345992106163812}{204848059751598401563305907296432335323118859258712413}$$

of τ satisfies $q_{112} > 6M$ and that $\varepsilon = ||q\mu|| - M||q\tau|| = 0.105822 > 0$. This implies, with $(A, B) = (175, \alpha)$ or $(6, \gamma)$, that either

$$n - n_1 \leqslant 271, \qquad \text{or} \qquad m - m_1 \leqslant 144.$$

We now look at each one of these two cases. First, we assume that $n - n_1 \leqslant 271$ and $m - m_1 \geqslant 20$. In this case, we consider

$$\Gamma_1 = n_1 \log \alpha - m \log \gamma + \log\left(\frac{4(\alpha^{n-n_1} - 1)}{\sqrt{10}}\right).$$

As before, $e^{\Gamma_1} - 1 = \Lambda_1 \neq 0$, so $\Gamma_1 \neq 0$. We go to (**10**). With an argument similar to a previous one, we have that

$$0 < |\Gamma_1| < \frac{2\gamma^4}{\gamma^{m-m_1}}.$$

Dividing through by $\log \gamma$ we obtain

$$0 < |n_1 \tau - m + \mu| < \frac{78}{\gamma^{m-m_1}},$$

where τ is the same one as above and

$$\mu := \frac{\log\left(4(\alpha^{n-n_1} - 1)/\sqrt{10}\right)}{\log \gamma}.$$

We apply again Lemma 3 noting that $n_1 > 0$, for otherwise we would have that $n \leqslant 271$ which contradicts our hypothesis that $n > 300$. Consider

$$\mu_k := \frac{\log\left(4(\alpha^k - 1)/\sqrt{10}\right)}{\log \gamma}, \qquad \text{for} \qquad k = 1, \ldots, 271.$$

We ran a *Mathematica* program and found that the same convergent p_{112}/q_{112} satisfies $q_{112} > 6M$. Further, $\varepsilon_k \geq 0.00119532$ for all $1 \leqslant k \leqslant 271$. For each of the values of ε_k and with $(A, B) = (78, \gamma)$, we calculate $\log(78q_{112}/\varepsilon_k)/\log\gamma$ and found that each of them is at most 152. Thus, $m - m_1 \leqslant 152$.

Now let us look at the other case. Assume that $m - m_1 \leqslant 144$ and $n - n_1 \geqslant 20$. We consider

$$\Gamma_2 = n\log\alpha - m_1\log\gamma + \log\left(\frac{4}{\sqrt{10}(\gamma^{m-m_1}-1)}\right).$$

We note that $1 - e^{-\Gamma_2} = \Lambda_2 \neq 0$, so $\Gamma_2 \neq 0$. We go to (**11**). With an argument similar to one above, we obtain

$$0 < |\Gamma_2| < \frac{2\alpha^7}{\alpha^{n-n_1}}.$$

Dividing through by $\log\lambda$, we get

$$0 < |n\tau - m_1 + \mu| < \frac{66}{\alpha^{n-n_1}},$$

where τ is the same one as above and

$$\mu := \frac{\log\left(4/(\sqrt{10}(\gamma^{m-m_1}-1))\right)}{\log\gamma}.$$

Now we use again Lemma 3 noting that $m_1 > 0$, which is the case, since otherwise we have $m \leqslant 144$, which contradicts our hypothesis $m > 163$. As above, by considering now

$$\mu_\ell := \frac{\log\left(4/(\sqrt{10}(\gamma^{\ell}-1))\right)}{\log\gamma}, \qquad \text{for all} \qquad \ell = 1, \ldots, 144$$

and running a *Mathematica* program, we find that $q_{112} > 6M$, and that for this convergent $\varepsilon_\ell \geq 0.0000620747$ for all $1 \leqslant \ell \leqslant 144$. For each of these ε_ℓ and with $(A, B) := (66, \alpha)$, we calculated $\log(66q_{112}/\varepsilon_\ell)/\log\alpha$ and found that all these numbers are at most 156. Thus $n - n_1 \leqslant 156$.

So, we got that either $n - n_1 \leqslant 271$ or $m - m_1 \leqslant 144$. Assuming the first one we deduced $m - m_1 \leqslant 152$, and assuming the second one, we deduced $n - n_1 \leqslant 156$. Altogether, we have $n - n_1 \leqslant 271$, $m - m_1 \leqslant 152$. So, it remains to study this case. We consider

$$\Gamma_3 = n_1\log\alpha - m_1\log\gamma + \log\left(\frac{4}{\sqrt{10}}\left(\frac{\alpha^{n-n_1}-1}{\gamma^{m-m_1}-1}\right)\right).$$

We note that $e^{\Gamma_3} - 1 = \Lambda_3$. Again, since $\Lambda_3 \neq 0$, we have that $\Gamma_3 \neq 0$. Since $n > 300$, we get

$$0 < |\Gamma_3| < \frac{2\alpha^8}{\alpha^n}.$$

Dividing through by $\log\gamma$, we get

$$0 < |n_1\tau - m_1 + \mu| < \frac{107}{\alpha^n},$$

where τ is as above and

$$\mu := \frac{\log\left(4(\alpha^{n-n_1}-1)/\sqrt{10}(\gamma^{m-m_1}-1)\right)}{\log\gamma}.$$

We apply for the last time Lemma (3). As above, we have that $n_1, m_1 > 0$. Thus, we consider

$$\mu_{k,\ell} := \frac{\log\left(4(\alpha^{k}-1)/\sqrt{10}(\gamma^{\ell}-1)\right)}{\log\gamma}, \quad k = 1,\ldots,271, \quad \ell = 1,\ldots,152.$$

Running a *Mathematica* program, we find again that the same convergent works namely $q_{112} > 6M$ and $\varepsilon_{k,\ell} \geq 0.0000307768$ for all $1 \leqslant k \leqslant 271$ and $1 \leqslant \ell \leqslant 152$. For each of these values $\varepsilon_{k,\ell}$, with $(A, B) := (107, \alpha)$, we calculated $\log(107q_{112}/\varepsilon_{k,\ell})\,/\log\alpha$ and found that the maximum value of them is $\leqslant 157$. Thus, $n \leqslant 157$, which contradicts our assumption on n.
This finishes the proof of our theorem.

Acknowledgements

F. L. was supported in part by grant CPRR160325161141 and an A-rated scientist award both from the NRF of South Africa and by grant no. 17-02804S of the Czech Granting Agency. F.L. worked on this paper during visits at the Mathematics Department to the UAZ, Zacatecas, Mexico and the Max Planck Institute for Mathematics, Bonn, Germany in February and March of 2018, respectively. He thanks these institutions for the generous support and hospitality. L.M.R. was partially supported by grant PFCE 2018-2019 (UAZ-CA-169).

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