ON THE k-TORSION OF THE MODULE OF DIFFERENTIALS OF ORDER n OF HYPERSURFACES

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ABSTRACT. We characterize the k-torsion freeness of the module of differentials of order n of a point of a hypersurface in terms of the singular locus of the corresponding local ring.

1. Introduction

The module of Kähler differentials of a ring is a classical object in commutative algebra. Recall that for a K-algebra R, the module of Kähler differentials is defined as the quotient $\Omega^1_{R/K} := I_R/I_R^2$, where I_R is the kernel of the multiplication map $R \otimes_K R \to R$. More generally, the module of Kähler differentials of order n can be defined as $\Omega^{(n)}_{R/K} := I_R/I_R^{n+1}$ (see, for instance, [8, 12, 13]).

It is well-known that the module of differentials can be used to detect properties of the ring. For instance, under some hypothesis, the regularity of the localization of a finitely generated algebra is equivalent to the freeness of its module of differentials. An analogous statement holds for the module of high order differentials (this was proved for hypersurfaces in [4] and, in a more general context, in [5]).

We are interested in studying other properties of certain rings that can be detected through its module of differentials. Let V be an affine variety over a perfect field $\mathbb{K}$. Suppose that V is locally, at some point $P \in V$, a complete intersection. Denote as R the corresponding local ring. It was proved by J. Lipman that V being non-singular at P in codimension 1 (resp. in codimension 2) is equivalent to the torsion freeness (resp. reflexiveness) of $\Omega^1_{R/\mathbb{K}}$ (see [10]). It was proved that the first statement of Lipman's theorem also holds for the module of high order differentials in the case of hypersurfaces (see [4]).

There is a general notion of k-torsion freeness for any $k \in \mathbb{N}$, $k \geq 1$, that generalizes the notions of torsion freeness and reflexiveness (see [3] or section 3 below). The main goal of this paper is to prove that k-torsion freeness of

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the module of high order differentials of a hypersurface can be characterized in terms of the singular locus.

Our approach to the problem is essentially the same as Lipman's. After making a careful analysis of his proof, we realized that part of the arguments were valid in a much more general situation. In addition, a key ingredient in Lipman's proof is the fact that the projective dimension of the module of differentials of a reduced locally complete intersection is less or equal than one. An analogous statement was proved in [4] for the module of high order differentials of hypersurfaces, allowing us to carry on with Lipman's strategy. Finally, the last ingredient we need for our proof is a criterion of regularity for hypersurfaces in terms of the module of high order differentials.

2. Modules of Kähler differentials

In this paper, all rings we consider are assumed to be commutative and with a unit element.

Let R be a K-algebra. Denote I_R the kernel of the homomorphism $R\otimes_K R\to R$, $r\otimes s\mapsto rs$. Giving structure of R-module to $R\otimes_K R$ by multiplying on the left, define the R-module

$$\Omega^{(n)}_{R/K} := I_R/I_R^{n+1}.$$

Definition 2.1. [13, Definition 1.5] The R-module $\Omega^{(n)}_{R/K}$ is called the *module of Kähler differentials of order n of R over K* or the *module of high order Kähler differentials.* For $n=1$, this is just the usual module of Kähler differentials of R.

A classical result states that, under some hypothesis, the localization of a finitely generated algebra R is regular if and only if $\Omega^1_{R/K}$ is free (see, for instance, [7, Chapter II, Theorem 8.8]). Another result in this direction is the following theorem due to J. Lipman (the statement (1) was also proved by S. Suzuki in [15]).

Theorem 2.2. [10, Proposition 8.1] *Let R be the local ring of a point P on an affine variety V over a perfect field $\mathbb{K}$. Assume that V is locally, at P, a complete intersection. Then*

(1) $\Omega^1_{R/\mathbb{K}}$ is torsion free if and only if V is non-singular in codimension 1 at P.
(2) $\Omega^1_{R/\mathbb{K}}$ is reflexive if and only if V is non-singular in codimension 2 at P.

In the statement of the theorem, *non-singular in codimension i at P* means that $\mathrm{codim}(R/\mathfrak{p})\geq i+1$, for all $\mathfrak{p}\in\mathrm{Sing}(R)$, where $\mathrm{codim}(R/\mathfrak{p}) = \dim R-\dim R/\mathfrak{p}$ and $\mathrm{Sing}(R)=\{\mathfrak{p}\in\mathrm{Spec}(R)|R_{\mathfrak{p}}\text{ is not regular}\}$.

The first statement of the theorem was generalized to the module of high order Kähler differentials of a hypersurface, following the strategy in [15].

Theorem 2.3. [4, Theorem 4.3] *Let R be the local ring of a point P on an irreducible hypersurface W over a perfect field $\mathbb{K}$. Then $\Omega^{(n)}_{R/\mathbb{K}}$ is torsion free if and only if W is normal at P.*

In the next section we recall the notion of k-torsion freeness of an R-module, for any positive integer k. If R is Noetherian and reduced, then the notions of torsion freeness and reflexiveness correspond, respectively, to 1-torsion freeness and 2-torsion freeness. Our main goal in this paper is to generalize Theorem 2.3 to apply to k-torsion freeness, for any $k \geq 1$.

3. A general theorem on k-torsion freeness

In this section we recall the notion of k-torsion freeness of a module. Then we give a characterization of this notion for modules having projective dimension less or equal than 1.

Let R be a Noetherian ring and let M be an R-module. The dual of M, denoted by M^*, is the module $\mathrm{Hom}_R(M,R)$. The bidual of M is denoted by M^{**}. The bilinear map $\phi : M \times M^* \to R$ defined by $\phi(m,\varphi) = \varphi(m)$ induces an R-homomorphism $f : M \to M^{**}$, given by $f(m) = \phi(m,\cdot)$. For a given R-homomorphism $\varphi : M \to N$, we denote as φ^* the induced map $N^* \to M^*$.

Let us suppose that M is a finite R-module, i.e., M is finitely generated. Since R is Noetherian, M is finitely presented, i.e., there exists an exact sequence

$$P_1 \xrightarrow{\varphi} P_0 \to M \to \mathbf{0},$$

where P_0, P_1 are finite free R-modules. Let $D(M) := \mathrm{Coker}(\varphi^*)$, which is known as the Auslander transpose of M. In [2] it is shown that the previous sequence induces the following exact sequence:

$$\mathbf{0} \to \mathrm{Ext}^1_R(D(M),R) \to M \xrightarrow{f} M^{**} \to \mathrm{Ext}^2_R(D(M),R) \to \mathbf{0}. \tag{1}$$

It is proved in [3] that for any $i \in \mathbb{N}$, $\mathrm{Ext}^i_R(D(M),R)$ depends only on M and not on the particular presentation $P_1 \to P_0 \to M \to \mathbf{0}$, where P_0 and P_1 are projective R-modules.

Remark 3.1. Recall that an $R-$module M is *torsionless* if f is injective and that M is *reflexive* if f is an isomorphism. Let Q be the total quotient ring of R. Then M is called *torsion free* if the natural map $\theta : M \to M_Q$ is injective, where $M_Q := M \otimes_R Q$. It is known that $\ker(\theta) \subset \ker(f)$. Thus, the concept of torsionless implies the concept of torsion free. If R is Gorenstein and has no embedded primes then the concepts are equivalent (see [16, Theorem (A.1)]).

In view of (1), torsionless and reflexiveness are respectively equivalent to $\mathrm{Ext}^1_R(D(M),R) = \mathbf{0}$ and $\mathrm{Ext}^1_R(D(M),R) = \mathrm{Ext}^2_R(D(M),R) = \mathbf{0}$. This leads us to the following general notion of k-torsion freeness.

Definition 3.2. [3] Let $k \in \mathbb{N}$, $k \geq 1$. We say that the R-module M is *k-torsion free* if $\mathrm{Ext}^i_R(D(M), R) = \mathbf{0}$, for $i \in \{1, \ldots, k\}$.

We want to study the k-torsion freeness of modules having projective dimension less or equal than one. For that, we need to recall some results concerning the grade and depth of modules.

Let R be a Noetherian ring, M be a finite R-module and I be an ideal of R. Recall that the *grade of the ideal I over* M, denoted as $\mathrm{grade}(I, M)$, is the maximal size of a M-regular sequence in I. It is known that $\mathrm{grade}(I, M)$ can be computed in the following way (see [6, Theorem 1.2.5]):

$$\mathrm{grade}(I, M) = \min\{i \in \mathbb{N} : \mathrm{Ext}^i_R(R/I, M) \neq \mathbf{0}\}.$$

We also define $\mathrm{grade}(M) := \min\{i \in \mathbb{N} : \mathrm{Ext}^i_R(M, R) \neq \mathbf{0}\}$. In addition, for a local ring $(R, \mathfrak{m})$ we denote $\mathrm{depth}(M) := \mathrm{grade}(\mathfrak{m}, M)$. Then, by [6, Proposition 1.2.10] we have

$$\mathrm{grade}(M) = \mathrm{grade}(\mathrm{Ann}(M), R) = \min\{\mathrm{depth}(R_{\mathfrak{p}}) : \mathfrak{p} \in \mathrm{Supp}(M)\}. \tag{2}$$

We also need some facts regarding Cohen-Macaulay rings. If $(R, \mathfrak{m})$ is a finite-dimensional local Noetherian ring, then $\mathrm{ht}(\mathfrak{p}) \geq \mathrm{depth}(R) - \dim(R/\mathfrak{p})$. Moreover, if R is Cohen-Macaulay, from this inequality we deduce that

$$\mathrm{depth}(R_{\mathfrak{p}}) = \dim R - \dim(R/\mathfrak{p}) = \mathrm{codim}(R/\mathfrak{p}). \tag{3}$$

With these tools at hand, now we can give a characterization of k-torsion freeness for modules having projective dimension less or equal than one. This characterization is based on part of the proof of Lipman's theorem 2.2.

Lemma 3.3. *Let M be an R-module and Q be the total quotient ring of R. If $M_Q = \mathbf{0}$, then $\mathrm{Ext}^0_R(M, R) = \mathbf{0}$.*

Proof. Let $Z(R)$ be the set of zero divisors of R, i.e., $Z(R) = \cup_{P \in \mathrm{Ass}(R)} P$. As $M_Q = \mathbf{0}$, we have $M_P = \mathbf{0}$ for every $P \in \mathrm{Ass}(R)$, which implies $\mathrm{Ann}(M) \nsubseteq Z(R)$. Thus, there exists $x \in \mathrm{Ann}(M)$ such that $x \notin Z(R)$. Therefore $0 < \mathrm{grade}(\mathrm{Ann}(M), R) = \mathrm{grade}(M) = \min\{i \in \mathbb{N} : \mathrm{Ext}^i_R(M, R) \neq \mathbf{0}\}$. We conclude $\mathrm{Ext}^0_R(M, R) = \mathbf{0}$. □

Theorem 3.4. *Let R be a Noetherian local ring with total quotient ring Q. Let M be a finite R-module with a finite projective resolution*

$$\mathbf{0} \to P_1 \xrightarrow{\varphi} P_0 \to M \to \mathbf{0}.$$

Let k be a positive integer. Then M is k-torsion free if and only if $\mathrm{depth}(R_{\mathfrak{p}}) \geq k + 1$*, for any* $\mathfrak{p} \in \mathrm{Supp}(D(M))$*. Moreover, if R is Cohen-Macaulay, M is k-torsion free if and only if* $\mathrm{codim}(R/\mathfrak{p}) \geq k + 1$*, for any* $\mathfrak{p} \in \mathrm{Supp}(D(M))$.

Proof. Using the projective resolution of M and the definition of $\mathrm{Ext}^1_R(M, R)$, it follows that $\mathrm{Ext}^1_R(M, R) = D(M)$. As the functor $\mathrm{Ext}^1_R(\cdot, \cdot)$ commutes with localization, we obtain

$$\begin{aligned} D(M) \otimes_R Q &= \mathrm{Ext}^1_R(M, R) \otimes_R Q \\ &\cong \mathrm{Ext}^1_{R\otimes_R Q}(M \otimes_R Q, R \otimes_R Q) \\ &= \mathrm{Ext}^1_Q(M_Q, Q). \end{aligned}$$

Since Q is the total quotient ring of R, any non-unit of Q is a zero divisor of Q, so $\mathrm{depth}(Q) = 0$. Moreover $\mathrm{projdim}(M_Q) \leq \mathrm{projdim}(M) \leq 1$, the last inequality is by hypothesis. Using the Auslander-Buchsbaum formula

$$0 = \mathrm{depth}(Q) = \mathrm{projdim}(M_Q) + \mathrm{depth}(M_Q),$$

we conclude $\mathrm{projdim}(M_Q) = 0$, so M_Q is projective. It follows that $\mathbf{0} = \mathrm{Ext}^1_Q(M_Q, R) \cong D(M) \otimes_R Q = D(M)_Q$. By lemma 3.3, $\mathrm{Ext}^0_R(D(M), R) = \mathbf{0}$. It follows that M is k-torsion free if and only if $\mathrm{Ext}^i_R(D(M), R) = \mathbf{0}$ for every $i \in \{0, \ldots, k\}$.

On the other hand, $\mathrm{Ext}^i_R(D(M), R) = \mathbf{0}$ for every $i \in \{0, \ldots, k\}$ if and only if $\mathrm{grade}(D(M)) \geq k + 1$. By (2), $\mathrm{grade}(D(M)) \geq k + 1$ if and only if $\mathrm{depth}(R_p) \geq k + 1$ for every $\mathfrak{p} \in \mathrm{Supp}(D(M))$. If, in addition, R is Cohen-Macaulay, by (3), $\mathrm{depth}(R_\mathfrak{p}) \geq k + 1$ if and only if $\mathrm{codim}(R/\mathfrak{p}) \geq k + 1$ for every $\mathfrak{p} \in \mathrm{Supp}(D(M))$. □

4. A characterization of k-torsion freeness

Now we are ready to generalize Theorem 2.3 for any $k \geq 1$. Throughout this section we use the following notation:

- $\mathbb{K}$ is a perfect field.
- $A = \mathbb{K}[x_1, \ldots, x_s]/\langle f\rangle$, where $f \in \mathbb{K}[x_1, \ldots, x_s]$ is irreducible.
- $W = \mathrm{Spec}(A)$.
- R is the local ring of a closed point $P \in W$.

Our first goal is to describe the support of the module $\mathrm{Ext}^1_R(\Omega^{(n)}_{R/\mathbb{K}}, R)$ in terms of the singular locus of R. First we need the following criterion of regularity for hypersurfaces in terms of the module of differentials of high order.

Proposition 4.1. *Let $\mathfrak{p} \in W$. Then $A_\mathfrak{p}$ is a regular ring if and only if $\Omega^{(n)}_{A_\mathfrak{p}/\mathbb{K}}$ is a free $A_\mathfrak{p}$-module. In addition, in this case the rank of $\Omega^{(n)}_{A_\mathfrak{p}/\mathbb{K}}$ is $L - 1$, where $L = \binom{s-1+n}{s-1}$.*

Proof. The "only if" part is well-known and holds in full generality (see, for instance, [9, Section 4.2]). We include the proof here for the sake of completeness.

If $A_\mathfrak{p}$ is a regular ring then $\Omega^1_{A_\mathfrak{p}/\mathbb{K}}$ is free of rank $s - 1$. In addition, in this case, $\mathbf{S}^n(I_{A_\mathfrak{p}}/I^2_{A_\mathfrak{p}}) = I^n_{A_\mathfrak{p}}/I^{n+1}_{A_\mathfrak{p}}$, where $\mathbf{S}^n(\cdot)$ denotes the nth-symmetric product. It follows that $I^n_{A_\mathfrak{p}}/I^{n+1}_{A_\mathfrak{p}}$ is free of rank $\binom{s-1+n-1}{s-2}$. Using the exact sequences

$$0 \to I^n_{A_\mathfrak{p}}/I^{n+1}_{A_\mathfrak{p}} \to I_{A_\mathfrak{p}}/I^{n+1}_{A_\mathfrak{p}} \to I_{A_\mathfrak{p}}/I^n_{A_\mathfrak{p}} \to 0,$$

it follows by induction that $\Omega^{(n)}_{A_\mathfrak{p}/\mathbb{K}}$ is free of rank $L-1$.

Now assume that $\Omega^{(n)}_{A_\mathfrak{p}/\mathbb{K}}$ is a free $A_\mathfrak{p}$-module. We first show that the rank of this module is $L-1$. Let $\Omega^{(n)}_{W/\mathbb{K}}$ be the sheaf of Kähler differentials of order n of W. By the assumption, there exists an open subset $U \subset W$ such that $\mathfrak{p} \in U$ and $\Omega^{(n)}_{W/\mathbb{K}}|_U$ is free. In particular, $(\Omega^{(n)}_{W/\mathbb{K}}|_U)_\mathfrak{q} \cong \Omega^{(n)}_{A_\mathfrak{q}/\mathbb{K}}$ is a free $A_\mathfrak{q}$-module for all $\mathfrak{q} \in U$. Since W is irreducible, U is irreducible as well, and so the rank of $\Omega^{(n)}_{A_\mathfrak{q}/\mathbb{K}}$ is constant in U. Let $\mathfrak{q}' \in U \subset W$ be a non-singular point (it exists because W is irreducible and so U and the open subset of non-singular points of W are dense). By the "only if" part of the proposition, $\Omega^{(n)}_{A_{\mathfrak{q}'}/\mathbb{K}}$ is free of rank $L-1$. It follows that the rank of $\Omega^{(n)}_{A_\mathfrak{p}/\mathbb{K}}$ is also $L-1$.

Now we show that $\Omega^{(n)}_{A_\mathfrak{p}/\mathbb{K}}$ free implies that $A_\mathfrak{p}$ is regular. We can assume that $U = D(g) \cong \mathrm{Spec}(A_g)$ is a principal open set. Since A_g is commutative with unit, there exists $\mathfrak{m} \subset A_g$ a maximal ideal such that $\mathfrak{p} \subset \mathfrak{m}$ [1, Corollary 1.4]. Then $A_\mathfrak{p} \cong (A_g)_\mathfrak{p} \cong ((A_g)_\mathfrak{m})_\mathfrak{p} \cong (A_\mathfrak{m})_\mathfrak{p}$. Since $\mathfrak{m} \in U$, it follows that $\Omega^{(n)}_{A_\mathfrak{m}/\mathbb{K}}$ is free of rank $L-1$. By [4, Theorem 3.1], $\mathfrak{m}$ being a maximal ideal implies that $A_\mathfrak{m}$ is a regular ring . We conclude that $A_\mathfrak{p} \cong (A_\mathfrak{m})_\mathfrak{p}$ is also a regular ring. □

Remark 4.2. It was proved in [5, Theorem 10.2] that the previous criterion of regularity holds more generally for local domains $(R, \mathfrak{m}, \mathbb{K})$ with pseudo-coefficient field K such that $\mathrm{Frac}(R)$ is separable over K and $\mathbb{K}$ is perfect. In particular, it holds for arbitrary irreducible varieties. In addition, an algebraic proof of the second part of the proposition can also be deduced from [5, Proposition 2.20].

Lemma 4.3. *Let S be a Noetherian local ring and M a finite S-module such that* $\mathrm{projdim}(M) \leq 1$. *Then* $\mathrm{Ext}^1_S(M,S) = \mathbf{0}$ *if and only if M is free.*

Proof. Suppose that $\mathrm{Ext}^1_S(M,S) = \mathbf{0}$, so $\mathrm{Ext}^1_S(M,F) = \mathbf{0}$ for any S-free module F. As $\mathrm{projdim}(M) \leq 1$ there exists an exact sequence

$$\mathbf{0} \to F_1 \xrightarrow{\varphi} F_0 \to M \to \mathbf{0}, \tag{4}$$

where F_0 and F_1 are finite free S-modules. Thus,

$$\mathbf{0} = \mathrm{Ext}^1_S(M,F_1) = \mathrm{Coker}\begin{pmatrix} \mathrm{Hom}(F_0,F_1) & \to & \mathrm{Hom}(F_1,F_1) \\ f & \mapsto & f\varphi \end{pmatrix}. \tag{5}$$

Therefore, there exists $f \in \mathrm{Hom}(F_0,F_1)$ such that $f\varphi = \mathrm{id}_{F_1}$; then φ splits and $F_0 \cong M \oplus F_1$. Thus M is projective and since S is a Noetherian local ring we conclude that M is free.

The converse of this lemma is immediate, because M is projective if and only if $\mathrm{Ext}^1_S(M,N) = \mathbf{0}$ for every S-module N. □

The next corollary follows the line of the proof of [10, Proposition 5.2]. The crucial additions are proposition 4.1 and the fact that $\Omega^{(n)}_{R/\mathbb{K}}$ has projective dimension less or equal than 1.

Corollary 4.4. *With the established notation,*

$$\operatorname{Supp}(\operatorname{Ext}^1_R(\Omega^{(n)}_{R/\mathbb{K}}, R)) = \operatorname{Sing}(R).$$

Proof. Let $\mathfrak{p} \in \operatorname{Spec}(R)$ be such that $R_\mathfrak{p}$ is regular. Then $\Omega^1_{R_\mathfrak{p}/\mathbb{K}}$ is a free $R_\mathfrak{p}$-module and so the same is true for $\Omega^{(n)}_{R_\mathfrak{p}/\mathbb{K}}$. Since the module of differentials of high order commutes with localization ([12, Theorem II-9]), lemma 4.3 implies

$$\mathbf{0} = \operatorname{Ext}^1_{R_\mathfrak{p}}(\Omega^{(n)}_{R_\mathfrak{p}/\mathbb{K}}, R_\mathfrak{p}) = \left(\operatorname{Ext}^1_R(\Omega^{(n)}_{R/\mathbb{K}}, R)\right)_\mathfrak{p}.$$

This shows that $\operatorname{Supp}(\operatorname{Ext}^1_R(\Omega^{(n)}_{R/\mathbb{K}}, R)) \subset \operatorname{Sing}(R)$.

Now let $\mathfrak{p} \in \operatorname{Spec}(R)$ be such that $(\operatorname{Ext}^1_R(\Omega^{(n)}_{R/\mathbb{K}}, R))_\mathfrak{p} = \mathbf{0}$. This implies $\operatorname{Ext}^1_{R_\mathfrak{p}}(\Omega^{(n)}_{R_\mathfrak{p}/\mathbb{K}}, R_\mathfrak{p}) = \mathbf{0}$. By [4, Theorem 4.3], $\operatorname{projdim}(\Omega^{(n)}_{R/\mathbb{K}}) \leq 1$. Thus, by lemma 4.3, $\Omega^{(n)}_{R_\mathfrak{p}/\mathbb{K}}$ is a free $R_\mathfrak{p}$-module. On the other hand, by the correspondence of prime ideals in R and A, we have $R_\mathfrak{p} \cong A_\mathfrak{p}$. In particular, $\Omega^{(n)}_{A_\mathfrak{p}/\mathbb{K}}$ is a free $A_\mathfrak{p}$-module. By proposition 4.1, $A_\mathfrak{p}$ is a regular ring. Thus $R_\mathfrak{p}$ is regular and so $\operatorname{Sing}(R) \subset \operatorname{Supp}(\operatorname{Ext}^1_R(\Omega^{(n)}_{R/\mathbb{K}}, R))$. □

Theorem 4.5. *Let $k \geq 1$. Then $\Omega^{(n)}_{R/\mathbb{K}}$ is k-torsion free if and only if W is non-singular in codimension $k+1$ at P.*

Proof. As before, $\operatorname{projdim}(\Omega^{(n)}_{R/\mathbb{K}}) \leq 1$. Consider the following projective resolution of $\Omega^{(n)}_{R/\mathbb{K}}$:

$$\mathbf{0} \to F_1 \xrightarrow{\varphi} F_0 \to \Omega^{(n)}_{R/\mathbb{K}} \to \mathbf{0}.$$

Since R is Cohen-Macaulay, we can apply theorem 3.4 to obtain that $\Omega^{(n)}_{R/\mathbb{K}}$ is k-torsion free if and only if $\operatorname{codim}(R/\mathfrak{p}) \geq k+1$ for any $\mathfrak{p} \in \operatorname{Supp}(\operatorname{Coker}(\varphi^*))$. In addition, using the previous exact sequence we obtain $\operatorname{Ext}^1_R(\Omega^{(n)}_{R/\mathbb{K}}, R) = \operatorname{Coker}(\varphi^*)$. By corollary 4.4 we conclude that $\Omega^{(n)}_{R/\mathbb{K}}$ is k-torsion free if and only if $\operatorname{codim}(R/\mathfrak{p}) \geq k+1$ for any $\mathfrak{p} \in \operatorname{Sing}(R)$. □

Remark 4.6. Notice that the entire strategy to prove theorem 4.5 can also be used to generalize Lipman's theorem 2.2 for k-torsion, for any $k \geq 1$.

Remark 4.7. One of the key ingredients of the proof of Theorem 4.5 was the fact that $\operatorname{projdim}(\Omega^{(n)}_{R/\mathbb{K}}) \leq 1$, where R is a local ring of an irreducible hypersurface. If this fact were also true for reduced complete intersections, then exactly the same strategy would give the analogous statement of Theorem 4.5 in this case. In this regard, an explicit presentation of $\Omega^{(n)}_{R/K}$ was recently given in [4, Theorem 2.8] for any finitely generated K-algebra R. Using this presentation one could try to compute the projective dimension of $\Omega^{(n)}_{R/K}$, at least in some examples of complete intersections. Unfortunately, due to the size of the matrix giving the presentation, we did not succeed in computing any example for $n > 1$, even with the help of a (modest) computer.

Remark 4.8. Even though the main goal of this paper was to generalize theorem 2.3, the results presented in section 3 apply to more general modules satisfying, among other hypothesis, that their projective dimension is less or equal than one. Families of modules satisfying this hypothesis can be constructed as in [17, Remark 2.1], [11, Lemma 1], or [14, Proposition 1.6].

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