Phase Unwrapping using Chebyshev Polynomials

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ABSTRACT

Phase unwrapping is an intermediate step for interferogram analysis. The phase associated with an interferogram can be estimated using a curve mesh of functions. Each of these functions can be approximated by a linear combination of basis functions. Chebyshev polynomials in addition to being a family of orthogonal polynomials can be defined recursively. In this work a method for phase unwrapping using Chebyshev polynomials is proposed. Results show good performance when applied to synthetic images without noise and also to synthetic images with noise.

Keywords: Phase unwrapping, interferometry, curve fitting.

Interferometry is a set of methods widely used to measure physical magnitudes such as deformation, stress, temperature, etc.[1,2] in a non destructive and non invasive way. These magnitudes modulate a fringes pattern called interferogram which contains information of a system been studied or analyzed. A processing of the interferogram is necessary to recover this information.

Standard techniques for phase recovery such as Fourier based,[3] phase stepping[4] or regularization,[5–7] provide a non-continuous phase wrapped in the interval $(-\pi, \pi]$. This phase needs to be unwrapped as a step to carry out the measure process of physical magnitudes. It is common to find phase inconsistences or noise that can make the unwrapping process a difficult task. The application of path dependent algorithms[8] improves the unwrapping process but does not always provide reliable results. A robust alternative for many cases is the least-squares, described in matrix form by Hunt.[9] Other robust algorithm to find a solution in the presence of path-integral phase inconsistencies, by using the cosine transform, is that proposed by Ghiglia and Romero.[10] The methods above represent long processing time and computational complexity that make them inconvenient for many practical applications. When the phase is smooth, the time of processing can be shortened solving the phase unwrapping problem by using a linear combination basis functions,[1120] . In this paper we propose a method to unwrap phase using a linear combination of First Order Chebyshev Polynomials. The weights are described in a typical matrix formulation allowing the matrix inversion using direct methods.[12]

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2. RELATIONSHIP BETWEEN WRAPPED AND UNWRAPPED PHASES

Let φ_r and ϕ_r the wrapped and the unwrapped phase respectively, where $r = (x, y)$ is the vector in a discrete grid, the relationship between these two phases is

$$\varphi_r = \mathcal{W}\{\phi_r\} = \phi_r + 2\pi k_r \tag{1}$$

where $\mathcal{W}$ represents the wrapping operator and k_r a field of integers such that $\mathcal{W}\{\phi_r\} \in (-\pi, \pi]$. The value of φ_r represents the observed phase (wrapped) and ϕ_r the real unknown phase (unwrapped) to be determined. The phase discrete gradient field, $\Delta\varphi_r$, is defined as

$$\nabla\varphi_r = (\varphi_r - \varphi_s, \varphi_r - \varphi_t) \tag{2}$$

where $s = r - (1, 0)$ and $t = r - (0, 1)$ are contiguous horizontal and vertical sites respectively. We can also define the unwrapped discrete gradient field as $\nabla\phi_r = (\phi_r - \phi_s, \phi_r - \phi_t)$. If the sampling theorem is fulfilled for these two discrete phase fields, the problem of the recovery ϕ_r from φ_r can be properly solved. The sampling theorem establishes that the distance between two fringes must be more than two pixels (the phase difference between two fringes is 2π). For phase, the sampling theorem is reached if the phase difference between two pixels is less than π. This is

$$\|\nabla\phi_r\| < \pi. \tag{3}$$

If this condition is satisfied, we can establish:

$$\nabla\phi_r = \mathcal{W}\{\nabla\varphi_r\} = (\mathcal{W}\{\varphi_r - \varphi_s\}, \mathcal{W}\{\varphi_r - \varphi_t\}) \tag{4}$$

$\mathcal{W}\{\nabla\varphi_r\}$ can be obtained from the observed field. From this equation, we see that ϕ_r can be achieved by two-dimensional integration of the vector field $\mathcal{W}\{\nabla\varphi_r\}$. This can be carried out by using a least-squares approach.[13–15]

3. FIRST ORDER CHEBYSHEV POLYNOMIALS FOR PHASE UNWRAPPING

3.1 Basis functions for function approximation

Any function can be aproximated by a linear combination of n basis functions. Let $U = [a, b] \in \mathbb{R}$ and $\hat{f}(x)$ an approximation of a function $f : U \mapsto \mathbb{R}$ using a set of basis functions $T(i, x)$, then

$$\hat{f}(x) = \sum_{i=0}^{n} w_i T(i, x) = \mathbf{T}\mathbf{w} \tag{5}$$

where $\mathbf{T} = [T(0, x) \mid T(1, x) \mid \cdots \mid T(n, x)]$ y $\mathbf{w} = [w_0\ w_1\ \cdots\ w_n]$. Minimizing the norm of the residual ρ ,

$$\|\rho\| = \min_{\mathbf{w}} \|f(x) - \hat{f}(x)\| \tag{6}$$

we obtain the optimal $\mathbf{w}^*$. Figure 1 shows two approximations for a test function using first order Chebyshev Polynomials.

3.2 Curve mesh

A surface can be considered as a mesh of functions (a net of interconnected functions). Several ways for surface representation using function of two independent variables from a mesh of 2D functions have been proposed[16–19]
.

A curve mesh M (which it is a set of all 2D functions in $[a, b] \times [c, d] \in \mathbb{R}^2$) can be expressed as:

$$M = \bigcup_{j=1}^{n} \{(x, y_j, g_j(x)) \,|x \in [a, b]\} \cup \bigcup_{i=1}^{m} \{(x_i, y, h_i(y)) \,|y \in [c, d]\} \tag{7}$$

Figure 2 shows a curve mesh where a patch is created by using Poisson Equation (it is also possible by using any other similar method)

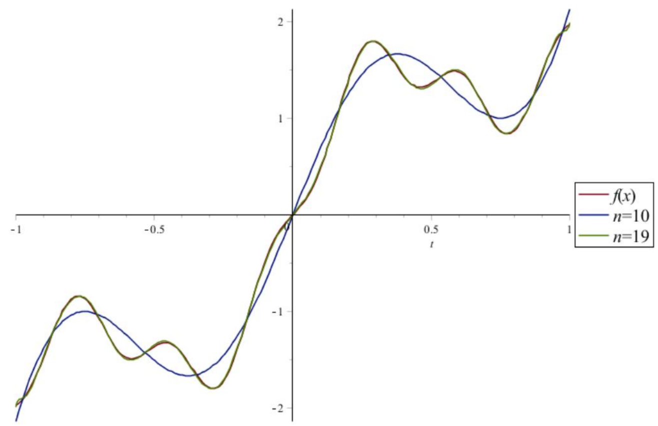


Figure 1. Function approximation using first order Chebyshev Polynomials.

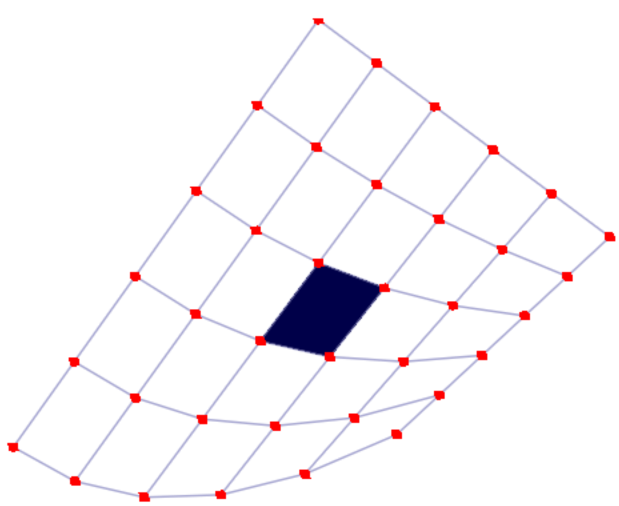


Figure 2. Function curve mesh representation.

3.3 Curve mesh for phase unwrapping

Let $\phi : V \mapsto \mathbb{R}$ a wrapped phase map and M its approximation by using a curve mesh, then

$$\phi\left(x_i, y\right) \approx h_i\left(y\right), \ \phi\left(x, y_j\right) \approx g_j\left(x\right) \tag{8}$$

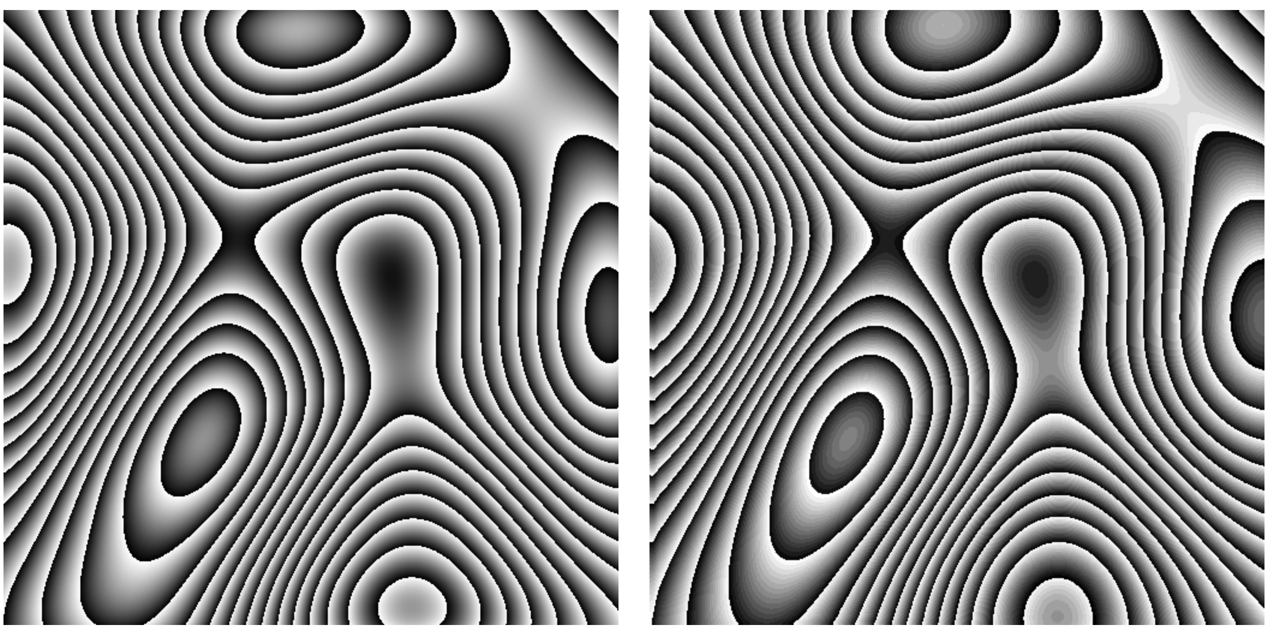
where $h_i\left(y\right)$ y $g_j\left(x\right)$ are approximated by Chebyshev polynomias. The unwrapped phase map, ϕ, is obtained by using

$$\min_{\mathbf{w}_i^x, \mathbf{w}_j^y} \|\rho\|^2 = \sum_{j=0}^{n} \|\mathcal{W}\left[\Delta_x \varphi\left(x, y_j\right)\right] - \Delta_x \hat{\phi}\left(x, y_j\right)\|^2 + \sum_{i=0}^{m} \|\mathcal{W}\left[\Delta_y \varphi\left(x_i, y\right)\right] - \Delta_y \hat{\phi}\left({}_i x, y\right)\|^2 \quad s.t.\ h_i\left(y\right) = g_j\left(x\right) \tag{9}$$

where $\mathbf{w}_i^x$, and $\mathbf{w}_j^y$ are optimal for the linear combination of Chebyshev polynomials for functions $g_j\left(x\right)$ and $h_i\left(y\right)$, respectively.

4. RESULTS

The proposed method was applied to synthetic phase map with and without noise. For each function , $g_j\left(x\right)$ and $h_i\left(y\right)$, 21 Chebyshev polynomials were used (from degree $n = 0$ to degree $n = 20$). Curve mesh, M, is formed by 50 horizontal and 50 vertical lines. Figure 3 shows the synthetic test wrapped phase map and its reconstruction (which is wrapped for comparison purposes). Figure shows the reconstruction but now for a synthetic noisy wrapped phase map.

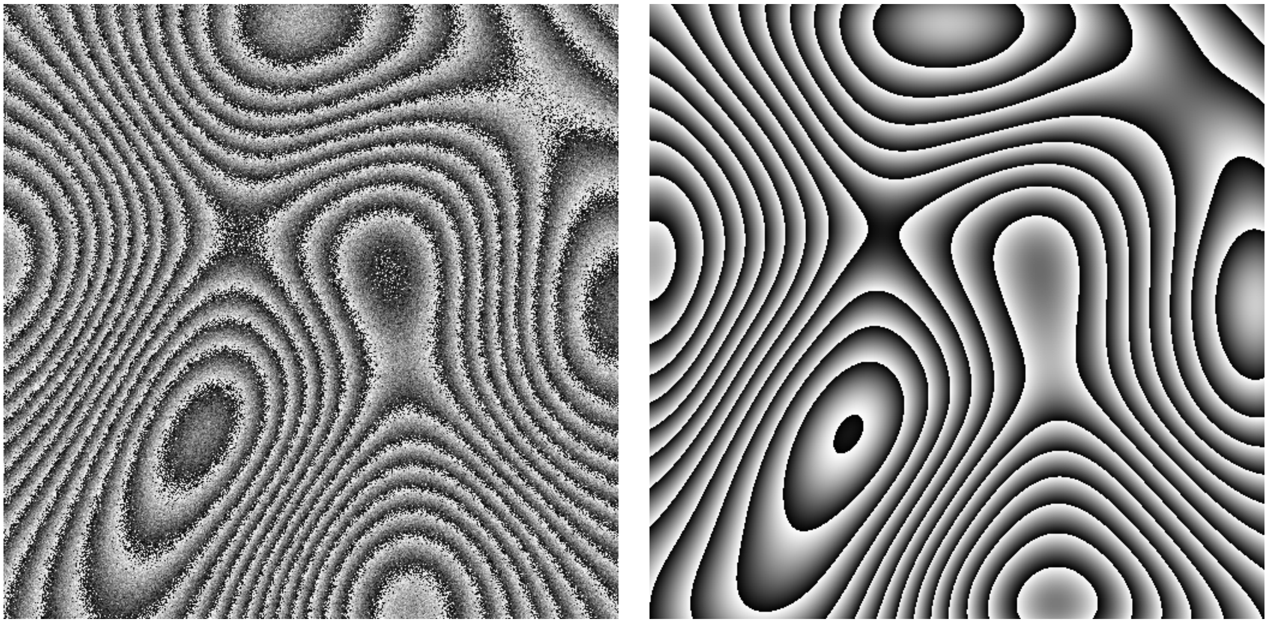


(a) (b)

Figure 3. Noiseless synthetic phase map (a) wrapped (b) approximated phase map by using Chebyshev polynomials.

5. CONCLUSIONS

We have presented an algorithm based on Chebyshev polynomials as basis functions to recover the phase from a wrapped phase map. This algorithm has good visual performance for smooth phase maps even with low level of noise. The sampling of the phase and the recuersive computation of Chebyshev polynomails allow decrease both the processing time and the memory resource required to unwrapped the phase.



(a) (b)

Figure 4. Noisy synthetic phase map (a)wrapped (b) phase map approximated by Chebyshev polynomials.

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