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2016 Eur. J. Phys. 37 045202

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The pedagogical value of the four-dimensional picture: III. Solutions to Maxwell's equations

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Received 26 October 2015, revised 25 January 2016

Accepted for publication 24 March 2016

Published 20 April 2016



CrossMark

Abstract

We outline a regular way for solving Maxwell's equations. We take, as the starting point, the notion of vector potentials. The rationale for introducing this notion in electrodynamics is that the set of Maxwell's equations is seemingly overdetermined. We demonstrate the existence of two fundamental solutions to Maxwell's equations whose linear combinations comprise the whole variety of classical electromagnetic field configurations.

Keywords: fundamental solutions of Maxwell's equations, gauge fields, vector potentials

1. Introduction

In this third paper of a series of papers, initiated by [1, 2], we continue to review the utility of four-dimensional concepts in classical electrodynamics.

The discussion of [2] made it clear that the law governing the electromagnetic field behavior is largely ordered by the geometry of Minkowski spacetime $\mathbb{R}_{1,3}$. This law is given by a system of partial differential equations

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$$\partial_\lambda F^{\lambda\mu} = 4\pi j^\mu, \quad (1)$$

$$\partial_\lambda {}^*F^{\lambda\mu} = 0, \quad (2)$$

known as Maxwell's equations. Here ∂_λ stands for $\partial/\partial x_\lambda$, and ${}^*F^{\lambda\mu} \equiv \frac{1}{2} \epsilon^{\lambda\mu\nu\rho} F_{\nu\rho}$. We use the Gaussian system of units, and put the speed of light to be 1. Fixing a particular inertial frame of reference, equations (1) and (2) can be rewritten as

$$\nabla \cdot \mathbf{E} = 4\pi \rho, \quad (3)$$

$$\nabla \times \mathbf{B} = 4\pi \mathbf{j} + \frac{\partial \mathbf{E}}{\partial t}, \quad (4)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (5)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \quad (6)$$

Although the set of differential equations in four-dimensional tensor form, equations (1)–(2), is mathematically equivalent to the set of differential equations in three-dimensional vector form, equations (3)–(6), the former is much more elegant than the latter.

We now address the issue of whether the four-dimensional covariant treatment of basic solutions to Maxwell's equations is favored over the corresponding three-dimensional vector treatment, which is adopted in modern textbooks on electrodynamics and used as a common practice for teaching these solutions at undergraduate level. The aim of this paper is to show that this is indeed the case. Our main concern is with two questions.

- i. Where did the notion of vector potentials come from?
- ii. Are there several fundamental solutions to Maxwell's equations such that their linear combinations form the whole variety of classical electromagnetic fields distributed over empty space?

The rationale, or at least a motivation, for introducing vector potentials is that the set of Maxwell's equations is seemingly overdetermined. A regular procedure for solving this set of differential equations is to express the electromagnetic field strength $F_{\mu\nu}$ in terms of vector potentials A_μ .

As to the second question, the answer is positive. In fact, there exist two fundamental solutions to Maxwell's equations whereby every electromagnetic field configuration can be constructed. We will see that the Liénard–Wiechert field and plane wave are acting as fundamental solutions of this kind in classical electrodynamics.

It will transpire that the four-dimensional covariant framework not only makes the analysis of the posed questions much easier, but also provides a decisive pedagogical insight into geometric and physical information which is encoded in Maxwell's equations.

2. The notion of vector potentials

At first glance the set of Maxwell's equations, equations (1)–(2), is *overdetermined*: eight equations are intended for finding six unknown functions $F^{\mu\nu}$. Matters can be improved by expressing the field strength $F^{\mu\nu}$ in terms of vector potentials A^μ ,

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \quad (7)$$

Recall that $F^{\mu\nu}$ is an antisymmetric tensor, so that the antisymmetric combination of two vectors ∂^μ and A^ν on the right of (7) is quite appropriate. With the ansatz (7), the second part

of Maxwell's equation, equation (2), is satisfied identically because $\epsilon^{\lambda\mu\nu\rho}\partial_\lambda\partial_\nu\equiv 0$. Substituting (7) into (1) gives

$$\square A^\mu - \partial^\mu\partial_\lambda A^\lambda = 4\pi j^\mu, \quad (8)$$

where $\square = \partial^\lambda\partial_\lambda = \partial^2/\partial t^2 - \nabla^2$ is the wave operator. We thus come to the set of equations, equation (8), which has the number of equations equal to the number of the functions sought.

Note that A^μ is defined in equation (7) up to adding the four-gradient of an arbitrary smooth scalar function $\partial^\mu\chi$. Indeed, the field strength $F^{\mu\nu}$ is unaffected by the replacement

$$A^\mu \rightarrow A'^\mu = A^\mu - \partial^\mu\chi. \quad (9)$$

These transformations of A^μ are called *gauge transformations*. We thus deal with the entire equivalence class of vector potentials related to each other by gauge transformations, rather than a concrete vector function. The term $\partial^\mu\chi$ in (9) is called the *gauge mode*. These modes do not contribute to the Lorentz force $qv^\nu F_{\mu\nu}$, and hence the dynamics of charged particles is unaffected by them. On the other hand, the current of charged particles j^μ is not the source of gauge modes. Indeed, it is clear that gauge modes satisfy equation (8) with the vanishing right-hand side of this equation, whence it follows that gauge modes are unaffected by j^μ , and their evolution is divorced from the evolution of the dynamical degrees of freedom described by $F_{\mu\nu}$.

This offers a clearer view of how the seemingly overdetermined set of partial differential equations becomes determined. The net dynamical degrees of freedom are augmented by the addition of auxiliary degrees of freedom, gauge modes, which equalizes the number of equations governing this extended field system to the number of field variables.

The corresponding treatment of Maxwell's equations in three-dimensional vector form, equations (3)–(6), is not as intelligible. Let us write components of A^μ in a particular inertial frame: $A^\mu = (\phi, \mathbf{A})$, or, equivalently, $A_\mu = (\phi, -\mathbf{A})$. Taking into account the definitions of the electric field $E_i = F_{0i}$ and the magnetic induction $B_i = -\frac{1}{2}\epsilon_{ijk}F^{jk}$ which were given in [2] we obtain from (7)

$$\mathbf{E} = -\frac{\partial\mathbf{A}}{\partial t} - \nabla\phi, \quad (10)$$

$$\mathbf{B} = \nabla \times \mathbf{A}. \quad (11)$$

It is then possible to verify, by inspection, that equations (10) and (11) provide a solution of equations (5) and (6). However, if the four-dimensional ansatz (7) was not taken as the starting point, then the three-dimensional ansatz (10)–(11) is an ingenious mathematical trick whose discovery is surprising.

The corresponding three-dimensional gauge transformations are

$$\phi \rightarrow \phi' = \phi - \frac{\partial\chi}{\partial t}, \quad (12)$$

$$\mathbf{A} \rightarrow \mathbf{A}' = \mathbf{A} + \nabla\chi. \quad (13)$$

It is unlikely that equations (10)–(11) and (12)–(13) might help to illuminate the origin and mathematical nature of ϕ and \mathbf{A} . Historically, the ansatz (11) was suggested by W Thomson who investigated the analogies of electric phenomena with those of elasticity, and by C Neumann, Weber and Kirchhoff in their studies on the induction of currents [3]. However, while on the subject of modern teaching in gauge field theory (specifically in electrodynamics) following its inner logic, and not according to its historical development, it

is apparent that the student should learn of the notion of vector potentials in the four-dimensional relativistic framework.

Equation (2) can be rearranged to give⁵

$$\partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} = 0. \quad (14)$$

We call equations (2) and (14) collectively the *Bianchi identity*. This name comes from the fact that if we adopt A_μ as the basic variables, then the possibility to express $F_{\mu\nu}$ in terms of A_μ , as shown in equation (7), becomes a synonym for making the left-hand sides of equations (2) and (14) identically vanishing.

Equation (8) cannot be solved *directly* because the differential operator $\Lambda_\lambda^\mu(\partial) = \delta^\mu_\lambda \square - \partial^\mu \partial_\lambda$ has no inverse⁶. To tackle this problem, we take advantage of the gauge arbitrariness. We impose a gauge fixing condition on A_μ , selecting a single representative of the equivalence class of vector potentials, to make the differential operator invertible. For example, if we choose the so-called Lorenz gauge condition

$$\partial_\lambda A^\lambda = 0, \quad (17)$$

then (8) becomes the inhomogeneous wave equation

$$\square A^\mu = 4\pi j^\mu. \quad (18)$$

This resolves the problem because the wave operator \square is invertible.

3. Fundamental solutions to Maxwell's equations

It would be difficult if not impossible to find in current texts on classical electrodynamics the statement that all feasible classical electromagnetic field configurations can proliferate through composing linear combinations of only *two* fundamental solutions to Maxwell's equations⁷. We intend to show that two electromagnetic field configurations, known as the Lienárd–Wiechert field and plane wave, may be regarded as fundamental solutions of this kind.

It is sufficient to restrict our consideration to A_μ generated by a single charged particle moving along a smooth timelike world line $z^\mu(s)$. We can readily extend this analysis to cover the case of several charged particles by taking the the sum of all such A_μ generated by their respective individual sources.

⁵ To see this, let us note that $\partial_\lambda {}^*F^{\lambda\rho} = \frac{1}{2} \epsilon^{\lambda\rho\mu\nu} \partial_\lambda F_{\mu\nu}$ is proportional to the sum of terms stemming from the antisymmetrization of $\partial_\lambda F_{\mu\nu}$. Among them, the plus signs have terms which can be represented as cyclic permutations of indices of $\partial_\lambda F_{\mu\nu}$, that is, $\partial_\lambda F_{\mu\nu}$, $\partial_\nu F_{\lambda\mu}$ and $\partial_\mu F_{\nu\lambda}$, while the terms of another triplet $\partial_\lambda F_{\nu\mu}$, $\partial_\nu F_{\lambda\mu}$ and $\partial_\mu F_{\nu\lambda}$ are assigned the minus sign. However, both triplets actually contain identical terms because $F_{\mu\nu} = -F_{\nu\mu}$. It follows that equations (2) and (14) are equivalent.

⁶ The expert reader will recognize that $\det \Lambda = \lambda_0 \lambda_1 \lambda_2 \lambda_3$ where $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ are eigenvalues of the operator $\Lambda_\lambda^\mu(k) = k^2 \delta^\mu_\lambda - k^\mu k_\lambda$, that is, solutions to the eigenvalue problem

$$\Lambda_\lambda^\mu \Psi_a^\lambda = \lambda_a \Psi_a^\mu, \quad (15)$$

where a runs from 0 to 3 (there is no summation over a in the right-hand side). It is clear from

$$\Lambda_\lambda^\mu(k) k^\lambda = 0 \quad (16)$$

that some eigenvalue (associated with the eigenvector Ψ^μ proportional to k^μ) is zero, hence $\det \Lambda = 0$.

⁷ To illustrate we refer to several popular textbooks [4–8] where this fact went unnoticed.

Thus, our concern is with finding exact solutions to the equation

$$\square A^\mu(x) = 4\pi q \int_{-\infty}^{\infty} ds v^\mu(s) \delta^4[x - z(s)], \quad (19)$$

in which $A^\mu(x)$ is the unknown variable, q is the charge of the particle, and $v^\mu = dz^\mu/ds$ its four-velocity. Since the partial differential equation (19) is linear in $A^\mu(x)$, its solution is written as the sum of some particular solution of this equation and general solution of the associated homogeneous equation

$$\square A^\mu = 0. \quad (20)$$

What is the most appropriate form of the particular solution to equation (19) for the description of the classical electromagnetic picture? The commonly accepted point of view is that the *retarded* vector potential A_{ret}^μ , called the Liénard–Wiechert potential, is just this solution. The procedure of derivation of the Liénard–Wiechert potential is outlined in every textbook. We thus only recall the form of this solution using the condensed four-dimensional Dirac notations [9]. Let x^μ be some point outside the world line $z^\mu(s)$. Define the lightlike four-vector $R^\mu = x^\mu - z^\mu(s_{\text{ret}})$ drawn from a point $z^\mu(s_{\text{ret}})$ on the world line, where the signal is emitted, to the point x^μ , where the signal is received. It is seen from figure 1 that R^μ is opposed to a ray of the past light cone with the vertex at x^μ . Consider the unit vector v^μ tangent to the curve $z^\mu(s)$ at the instant s_{ret} and define the scalar

$$\rho = R_\alpha v^\alpha. \quad (21)$$

Since R^μ is a lightlike vector, the geometric interpretation of ρ is apparent: ρ is the spatial distance between the field point and the retarded point in the instantaneously comoving Lorentz frame in which the charge is at rest at the retarded instant s_{ret} , as shown in figure 2.

The retarded vector potential due to a single arbitrarily moving charge q is

$$A_{\text{ret}}^\mu(x) = q \frac{v^\mu}{\rho}. \quad (22)$$

Note that this expression for the retarded vector potential can be directly derived from that for the Coulomb potential, with the understanding that the retardation condition is met, see, e.g., [5].

The strength of the Liénard–Wiechert field is readily calculated from (22) with the aid of simple differentiation rules (these rules can be found in the textbooks [6–8]). The result is⁸

$$F_{\text{ret}}^{\mu\nu} = q \frac{R^\mu U^\nu - R^\nu U^\mu}{\rho^3}, \quad (23)$$

$$U^\mu = (1 - a_\alpha R^\alpha) v^\mu + \rho a^\mu, \quad (24)$$

where $a^\mu = dv^\mu/ds$ is the four-acceleration of the charged particle.

It might be well to point out that the four-dimensional description represents the Liénard–Wiechert field, equations (23)–(24), in a concise and elegant form. In contrast, the conventional three-dimensional vector treatment leads to rather cumbersome expressions

$$\mathbf{E}_{\text{ret}} = \frac{q}{(r - \mathbf{r} \cdot \mathbf{v})^3} \{ (1 - v^2)(\mathbf{r} - r\mathbf{v}) + \mathbf{r} \times [(\mathbf{r} - r\mathbf{v}) \times \mathbf{a}] \}, \quad (25)$$

⁸ A diligent student will perform this calculation as a useful exercise.

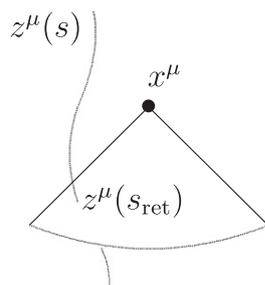


Figure 1. Retarded signal at x^μ , emitted from $z^\mu(s_{\text{ret}})$.

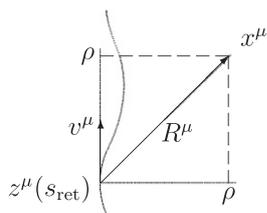


Figure 2. The origin of the invariant retarded variable ρ .

$$\mathbf{B}_{\text{ret}} = \frac{\mathbf{r} \times \mathbf{E}_{\text{ret}}}{r}, \quad (26)$$

where \mathbf{r} is the radius vector drawn from the point of emission $\mathbf{z}(t_{\text{ret}})$ to the point of observation \mathbf{x} in a particular Lorentz frame.

Let a particle be moving along a straight line $z^\mu(s) = z^\mu(0) + V^\mu s$, $V^\mu = \text{const}$. Then, in a Lorentz frame in which the time axis is parallel to V^μ , we have $U^\mu = V^\mu$, and equations (23)–(24) describe the Coulomb field. In a sense this feature remains valid for the field generated by an arbitrarily moving charge. Indeed, substituting equations (23)–(24) into the expressions for the electromagnetic field invariants

$$\mathcal{S} = \frac{1}{2} F_{\mu\nu} F^{\mu\nu}, \quad (27)$$

$$\mathcal{P} = \frac{1}{2} F_{\mu\nu} {}^*F^{\mu\nu} \quad (28)$$

gives

$$\mathcal{S}_{\text{ret}} = -\frac{q^2}{\rho^4}, \quad \mathcal{P}_{\text{ret}} = 0. \quad (29)$$

Since $\mathcal{S} = \mathbf{B}^2 - \mathbf{E}^2$, and $\mathcal{P} = -2\mathbf{E} \cdot \mathbf{B}$, this result implies that, whatever smooth world line is chosen, one can find such a frame of reference (which is peculiar to every point x^μ) that $\mathbf{B}_{\text{ret}} = 0$ and $|\mathbf{E}_{\text{ret}}| = q/r^2$ in all points of spacetime, that is, only electric field is observed in this frame. Thus, there exists a global (noninertial) frame of reference in which the retarded electromagnetic field generated by a single arbitrarily moving charge, shown in equations (23)–(24), appears as a pure Coulomb field at each observation point.

The Liénard–Wiechert field (23) is determined not only by the field F_{ret} as such but also by the frame of reference in which F_{ret} is measured. If we are to identify the net degrees of freedom related to F_{ret} , irrespective of the used frame of reference, we conclude that just the Coulomb field is responsible for those degrees of freedom.

This inference may seem surprising: any charged particle generates a field of electric type! However, we are well aware of the fact that not only fields of electric type but also fields of magnetic type are available in nature. Where do they come from? One can indicate at least two origins of magnetic fields. First, the superposition principle. It is an easy matter to verify from equations (23)–(24) that the relations $\mathcal{P} = 0$, $\mathcal{S} < 0$ are in general no longer valid for electromagnetic fields generated by two or several charges, so that the configurations involved may well represent fields of magnetic type. Note that the occurrence of pure magnetic fields due to the circulation of electrons around closed paths suggests a neutral system where electric fields of moving electrons and immovable nuclei mutually cancel. Second, a pure magnetic field may also be related to spin and its associated magnetic dipole moment of the charged particle, which, however, is beyond the scope of the present discussion.

The general solution of the homogeneous wave equation (20) can be written as an arbitrary superposition of plane waves $\epsilon_\mu \exp(ik \cdot x)$ with a lightlike propagation vector, k^μ , $k^2 = 0$, and the polarization vector ϵ_μ orthogonal to the propagation vector, $\epsilon \cdot k = 0$. Since our interest here is only with fields distributed over empty space, it is adequate to use a Fourier-integral expansion, so that the desired solution is

$$A^\mu(x) = \int \alpha(k) \epsilon^\mu(k) e^{ik \cdot x} d^4k. \quad (30)$$

We thus have established our assertion concerning the existence of two fundamental solutions to Maxwell’s equations giving rise to the variety of field configurations in classical electrodynamics, with due reservation of course that all the field configurations in macroscopic media were left aside in the present consideration.

4. Discussion and outlook (for the expert reader)

Since the discovery of the Aharonov–Bohm effect⁹, the quantity A_μ achieved settled status of the basic variable for accounting of the electromagnetic field in quantum theory¹⁰. Most current theories in high energy physics and gravity begin with gauge invariance as a first principle, that is, proceeding from vector potentials as the basic field variables. To illustrate, we take a glance at the Yang–Mills–Wong theory [13, 14]. The dynamical equations governing the Yang–Mills field read

$$\partial_\mu G_a^{\mu\nu} + g f_{abc} A_\mu^b G_{\mu\nu}^c = 4\pi j_a^\nu, \quad (31)$$

where $G_a^{\mu\nu}$ is the non-Abelian field strength which is expressed in terms of vector potentials A_μ^b as

$$G_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + g f_a^{bc} A_b^\mu A_c^\nu, \quad (32)$$

g is the Yang–Mills coupling constant, f_{abc} is the structure constants of the gauge group involved, and j_a^μ is the color charge current of N point particles, each carrying the color

⁹ This effect would be more properly termed the Ehrenberg–Siday–Aharonov–Bohm effect because it was discovered by Ehrenberg and Siday 10 years before Aharonov and Bohm [10, 11].

¹⁰ Note, however, that this status of vector potentials was challenged in [12]. We will not go into the details of this controversial issue, and refer the interested reader to the original literature.

charge Q_I^a , analogous to the current of N electrically charged point particles,

$$j_a^\mu(x) = \sum_{I=1}^N \int_{-\infty}^{\infty} ds_I Q_{aI}(s_I) v_I^\mu(s_I) \delta^4[x - z_I(s_I)]. \quad (33)$$

Let T_a be the generators of the gauge group. All color variables (the field strength, vector potentials, color charges, etc) can be written in matrix notation, as exemplified by $G_{\mu\nu} = \frac{i}{g} T_a G_{\mu\nu}^a$. Then equations (31) and (32) become

$$[D_\mu, G^{\mu\nu}] = 4\pi j^\nu, \quad (34)$$

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig [A_\mu, A_\nu]. \quad (35)$$

Here, the square brackets stand for commutators of matrix-valued quantities, and D_μ is the so-called covariant Yang–Mills derivative whose action on any field $\phi = \phi^a T_a$, transforming according to the adjoint representation of the gauge group, is given by

$$D_\mu \phi = \partial_\mu \phi + g [A_\mu, \phi]. \quad (36)$$

Note that for any gauge covariant quantity ϕ ,

$$[D_\mu, D_\nu] \phi = g [G_{\mu\nu}, \phi]. \quad (37)$$

By recognizing that $G_{\mu\nu}$ is expressed in terms of A_μ , according to (35), we come to a condition underlying this relation, the Bianchi identity,

$$[D_\lambda, G_{\mu\nu}] + [D_\nu, G_{\lambda\mu}] + [D_\mu, G_{\nu\lambda}] = 0, \quad (38)$$

which can be verified through the use of the Jacobi identity

$$[D_\lambda, [D_\mu, D_\nu]] + [D_\nu, [D_\lambda, D_\mu]] + [D_\mu, [D_\nu, D_\lambda]] = 0, \quad (39)$$

combined with equation (37).

Of course if we define the field strength $G_{\mu\nu}^a$ in terms of vector potentials A_μ^a according to (35), then there is no need to join the Bianchi identity (38) to the field equation (34) to complete the dynamics of this theory. Since the gauge-dependent quantities A_μ^a appear in the Yang–Mills theory from the outset, the set of dynamical equations is no longer over-determined. The situation with the Bianchi identity in general relativity closely resembles that in the Yang–Mills theory [15].

We thus see that classical electrodynamics offers a very instructive example of how the concept of gauge fields and gauge invariance can be introduced in their simplest physical and mathematical context.

One further feature of classical electrodynamics is its remarkably transparent field configuration arrangement: every field configuration stems from the Coulomb fields and plane waves. Note, however, that our concern here is with the fundamental aspects, rather than practical uses, of this arrangement. The Fourier-integral expansion (30) is sound but not universally convenient. In some instances it would be appropriate to expand the solution of the homogeneous wave equation (20) in terms of spherical harmonics. For example, a plausible guess about the nature of ball lighting is that the essential prerequisite to ball lighting formation is a steady-state field generated by converging and diverging axially symmetric microwaves [16, 17]. A self-dual solution to free Maxwell's equations [18] may be suitable to the analysis of standing wave configurations in this as yet unsolved problem.

By contrast, exact solutions in the classical Yang–Mills theory and general relativity pose many problems. The dynamical equations of these theories are nonlinear, so that any superposition of solutions appears to be something other than a new solution. Solutions to the classical Yang–Mills equations, known by the end of the 1970s, are reviewed in [19]. Exact

solutions of quantum Yang–Mills theory are altogether out of the question. This task is among one of the seven problems recorded by the Clay Mathematics Institute as the Millennium Prize Problems—the most difficult issues with which mathematicians were struggling at the turn of the second millennium [20]. A large body of exact solutions in general relativity are systematized in the catalog [21].

Are there exact solutions of these theories similar to the Liénard–Wiechert solution (24)? Such solutions to the Yang–Mills equations were indeed found in [22] (for a review see [23]), and rediscovered in [24]. These solutions fall into two classes. One of them contains solutions describing Yang–Mills fields of electric type, whose field invariants \mathcal{P} and \mathcal{S} built out of $G_{\mu\nu}^a$ are $\mathcal{P} = 0$ and $\mathcal{S} < 0$, while the other contains solutions describing fields of magnetic type specified by $\mathcal{P} = 0$ and $\mathcal{S} > 0$. These two types of solutions are very likely related to two phases of the Yang–Mills vacuum.

As to general relativity, exact solutions describing the gravitational field generated by an arbitrarily moving massive particle, similar to the Liénard–Wiechert solution, still remain unknown.

The non-Abelian analogues of electromagnetic plane waves, that is, exact solutions to equation (34) with $j^\mu = 0$ obeying the requirements that the energy density is bounded throughout spacetime, the direction of the Poynting vector is constant, and the magnitude of the Poynting vector is equal to the energy density, were found in [25]. However, such waves moving in different directions cannot be superposed, and hence, they are of no practical importance.

References

- [1] Kosyakov B P 2014 The pedagogical value of the four-dimensional picture: I. Relativistic mechanics of point particles *Eur. J. Phys.* **35** 025012
- [2] Kosyakov B P 2014 The pedagogical value of the four-dimensional picture: II. Another way of looking at the electromagnetic field *Eur. J. Phys.* **35** 025013
- [3] Whittaker E 1910 *A History of the Theories of Aether and Electricity. The Classical Theories* (London: Nelson) (2nd revised and enlarged edn 1951, p 242)
- [4] Jackson J D 1962 *Classical Electrodynamics* (2nd edn 1975, 3rd edn 1999) (New York: Wiley)
- [5] Landau L D and Lifshitz E M 1980 *The Classical Theory of Fields* (New York: Butterworth-Heinemann)
- [6] Synge J L 1956 *Relativity: The Special Theory* (Amsterdam: North Holland)
- [7] Barut A O 1964 *Electrodynamics and Classical Theory of Fields and Particles* (New York: Collier-Macmillan) (2nd edn 1980 (New York: Dover))
- [8] Rohrlich F 1965 *Classical Charged Particles* (Reading, MA: Addison-Wesley) (2nd edn 1990, 3rd edn 2007 (Singapore: World Scientific))
- [9] Dirac P A M 1938 Classical theory of radiating electron *Proc. R. Soc. A* **167** 148–69
- [10] Ehrenberg W and Siday R E 1949 The refractive index in electron optics and the principles of dynamics *Proc. Phys. Soc. B* **62** 8–21
- [11] Aharonov Y and Bohm D 1959 Significance of electromagnetic potentials in quantum theory *Phys. Rev.* **115** 485–91
- [12] Vaidman L 2012 Role of potentials in the Aharonov–Bohm effect *Phys. Rev. A* **86** 040101
- [13] Yang C N and Mills R 1954 Conservation of isotopic spin and isotopic gauge invariance *Phys. Rev.* **96** 191–5
- [14] Wong S K 1970 Field and particle equation for the classical Yang–Mills field and particles with isotopic spin *Nuovo Cimento A* **65** 689–94
- [15] Misner C W, Thorne K S and Wheeler J A 1973 *Gravitation* (San Francisco, CA: Freeman) pp 334–80
- [16] Kapitza P L 1955 On the nature of ball lightning *Dokl. Acad. Nauk SSSR* **101** 245–8
- [17] Stenhoff M 1999 *Ball Lightning: An Unsolved Problem in Atmospheric Physics* (New York: Kluwer)

- [18] Chubykalo A E, Espinoza A and Kosyakov B P 2009 Self-dual electromagnetic fields *Am. J. Phys.* **78** 858–61
- [19] Actor A 1979 Classical solutions of SU(2) Yang–Mills theories *Rev. Mod. Phys.* **51** 461–525
- [20] Jaffe A and Witten E 1999 Quantum Yang–Mills theory *The Millennium Prize Problems* ed J Carlson *et al* (Cambridge: Clay Mathematics Institute) pp 129–52
- [21] Kramer D *et al* 1980 *Exact Solutions of the Einstein’s Field Equations* (Berlin: Deutscher Verlag der Wissenschaften)
- [22] Kosyakov B P 1991 *Teor. Mat. Fiz.* **87** 422–4
Kosyakov B P 1991 Field of arbitrarily moving colored charge *Theor. Math. Phys.* **87** 632–4 (Engl. transl.)
- [23] Kosyakov B P 1998 Exact solutions in the Yang–Mills–Wong theory *Phys. Rev. D* **57** 5032–48
- [24] Sarioğlu Ö 2002 Lienárd-Wiechert potentials of a non-Abelian Yang–Mills charge *Phys. Rev. D* **66** 085005
- [25] Coleman S 1977 Non-Abelian plane waves *Phys. Lett.* **70** 59–60