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# GAUGE SYMMETRIES AND GAUGE TRANSFORMATIONS FOR MAXWELL EQUATIONS

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## Abstract

In a previous publication [5] we introduced gauge invariant electromagnetic potentials. However the field equations that were derived for them turn out to be the Coulomb gauge field equations. This raises the question of the nature of these new gauge invariant potentials and its relations with Coulomb gauge potentials. In this paper we prove that this new potentials are in fact gauge invariant and identical with the Coulomb gauge potentials. In other words: the Coulomb gauge potentials are gauge invariant potentials that cannot be related to Lorenz gauge potentials because, contrary to current ideas, there is not anything like a gauge transformation relating both gauges.

## 1. Introduction

At the level of description of the world where Maxwell equations are meaningful, it is generally accepted on empirical grounds that signals propagate causally through the vacuum and that this causality is mathematically reconstructed with the help of retarded potentials arising from D'Alembert equations for the field strengths or the potentials.

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However when potentials are used for the task of giving a mathematical representation of the generally accepted world view we face the problem of solving a pair of highly complex coupled partial differential equations (pde's). Hence atrick is proposed: we choose a new differential condition to decouple the pde's. This differential condition is known as "gauge" and the generally accepted mathematical doctrine around this kind of conditions is that they are an arbitrary choice made on the basis of the gauge invariance of the coupled pair of differential equations. Of course, when we select a gauge we can find a pair of D'Alembert equations for the potentials (Lorenz gauge) or a D'Alembert equation for the vector potential and a Poisson equation for the scalar one (Coulomb gauge) or any other kinds of equations (see, e.g. [1, 2]). According to generally accepted wisdom only the Lorenz gauge is physically sound, because in that gauge the physically relevant retarded solutions arise, while the other gauges are just mathematical devices to obtain a solution, being all of them equivalent in the sense that a group of gauge transformations can be constructed to go from one to the other. So, the physically sound gauge can be obtained from any gauge using a gauge transformation. This is the doctrine that we can find on almost any textbook on classical electrodynamics. However, as Jackson readily admits (see [2] "... the textbooks rarely show explicitly the gauge function ...". A gap that Jackson's quoted papers are trying to fill.

There is an extensive literature on the subject matter of Maxwell equations space-time symmetries, where almost all philosophical, mathematical and physical aspects are worked in detail at every level of sophistication. However, as we have seen, its gauge symmetries are worth a brief mention in almost any textbook and if worked out in depth it is only in relation to gauge field theories like Yang-Mills in the framework of the standard model (see [1] for a good example of this and a neat Whig history of the subject) where the gauge transformations involved are more complex than the gauge transformations usually associated to Maxwell equations.

However gauge symmetries are not trivial at all, indeed they are an odd presence because overdetermine the system of differential equations in the sense that every boundary value problem is ill-posed due to the addition of an arbitrary continuous function (for its role in quantizing see, e.g. [3]). Overdetermination is a feature of any set of covariant and gauge invariant differential equations, being the Einstein equations a famous instance no less than Maxwell equations. But as we have pointed out previously it is not easy to find a treatment of the subject where a complete set of consistent differential conditions for a gauge transformation for Maxwell equations is given. It looks as if it were an obvious thing. Indeed most textbooks give such an idea: it is easy to find a gauge transformation to do any kind of job, substantiating the statement showing an easy to solve differential equation that the gauge function satisfies. This is what Kaku did in [3]. Jackson, in the cited references, is certainly clear about this point, offering a more detailed treatment.

This ideology is annoying when one tries to relate different gauges. A natural suspicion about the existence of a gauge transformation that can realize such a job arises because of the freedom to choose a set of field equations that gauge invariance allows: in the Coulomb gauge the field equation for the scalar potential is, in most of the domains, elliptic, while in the Lorenz gauge the scalar potential satisfies a hyperbolic equation. Hence the typical gauge transformation for the Maxwell equations must relate, by adding a continuous function, an elliptic equation solution to a hyperbolic equation solution. Even leaving aside the fact that a Cauchy problem for an elliptic equation is always ill-posed (see e.g. [4]) while this is not the case for an hyperbolic equation (i.e.: we can prescribe Cauchy data for the scalar potential in the Lorenz gauge while this is not generally possible for the scalar potential in the Coulomb gauge but we can still find a function that relates them) the task looks difficult, but it is very important for many fields of physics (see e.g. [1] and references therein). Therefore one would expect a rigorous proof of this proposition at hand in the wide market of mathematical physics. However such a proof is not easy to find. Even the Jackson treatment is not really complete and a detailed proof of the existence of a gauge transformation is missing: he starts from the idea that the gauge function exists and is known once a solution to an inhomogeneous D'Alembert equation is given.

This article emerges from the conviction that a more detailed treatment of Maxwell equations gauge symmetries is required, specially when different gauges are involved. For that reason, we try to make the theory of gauge transformations for Maxwell equations clear using the lagrangian formalism and its associated apparatus of jet bundles. Another related reason is the set of difficult matters arising from the proof [5] that using Helmholtz Theorem we can leave aside any gauge transformation. Specifically, as Onoochin and Engelhardt have shown, the differential equations deduced using Helmholtz Theorem are equivalent to those of the Coulomb gauge, but it is a generally accepted doctrine that Coulomb gauge potentials can be the object of a gauge transformation while the potentials introduced in [5] cannot. This is indeed a result that appears because there is not available a complete general treatment of Maxwell equations gauge symmetries. In this paper we shall show that Onoochin and Engelhardt are right, but that [5] is right too. That is: the use of the Helmholtz Theorem is just another representation of the Coulomb gauge, but one where one can prove that Coulomb gauge symmetries can be eliminated and the potentials are gauge invariant. Besides we are going to give a substantial answer to the question of the existence of a gauge transformation that relates the Coulomb and Lorenz gauges (see [5]).

To achieve the goals we have described we use the well known fact that Maxwell equations are lagrangian equations, therefore we can introduce the invariance ideas directly from the lagrangian formalism to work out its gauge symmetries leaving aside the space-time symmetries and the most general but cumbersome Lie methods for treating symmetries. Indeed, if for us the important concepts are those of *s*-equivalence and *s*-equivariance (see def. 3 and def. 4 below), the Lie methods are not really useful, because with them we can build symmetries, but when we go from one gauge to another is not a symmetry transformation what is involved. Hence, we speak of gauge symmetries and *s*-equivariance when the respective lagrangian is *s*-equivariant under a group of transformations that we can construct, a group that relates solutions of the field equations derived from the lagrangian. And we shall speak of gauge groups and *s*-equivalence when we try to relate different lagrangians with the help of a gauge group, a gauge group whose task is to relate solutions to different field equations; in our case the most important instance is a gauge transformation that relates a solution to a Poisson type equation to a solution to an inhomogeneous D'Alembert equation.

The organization of the paper is as systematic as we can, trying at every step to make our hypothesis clear. For this reason we have used definitions, lemmas and theorems to present our results, even if many of them are quite trivial or obvious: it is against obviousness that our assault is directed.

So in section II we present most of the definitions that we found useful: we say what a jet bundle is, what are to a section, an equivalence problem, the concepts of $s$-equivariance and $s$-equivalence to make the paper a bit self-contained. We believe that this is almost all the mathematical machinery we need. So in section III we leave generality to introduce the Maxwell lagrangian and the manifolds that Coulomb and Lorenz gauges define with its respective particular lagrangians. Finally, we solve the equivalence problems for both, showing its gauge groups by giving the explicit equations that the gauge function satisfies. All these things are more or less known except the proof that in the Coulomb gauge the potentials are gauge invariant; a result that helps us to clarify the use of the Helmholtz Theorem as applied in [5]. However with these results assured we may advance towards our main result: there is no gauge transformation connecting the Lorenz and Coulomb gauges. In the conclusions, we present our understanding of this result.

## 2. Mathematical Framework and Statement of the Problem

### 2.1. The 1-jet bundle

Let us take a triple $B(M) = \langle B, \pi, M \rangle$ as a fiber bundle i.e. a locally trivial fibred manifold such that the map $\pi : B \to M$ between the differentiable manifold $B$ and the differentiable base manifold $M$ has a surjective tangent map $\pi_*$ with split kernel on the $TB$, the tangent space to $B$ (see e.g. for general [6, 7, 8] and for short introductions [9, 10]). We shall suppose that there exist differentiable sections $f : M \to B$ of, $B(M)$, which, of course, satisfy $\pi^{\circ} f = id_M$. We use the symbol $S(B)$ for the space of sections of our fiber bundle. It is in the space of sections that we define an equivalence relation $R_p^1$ in the following way: take two sections $f, \gamma \in S(B)$ and say that both are $R_p^1$-related if and only if, there is a point $p \in M$ at whose neighborhood we have:

$$\gamma(p) = f(p)$$

$$\gamma_* = f_* \; all \; over \; T_p M$$

So we have a quotient space $S(B)/R_p^1$ for whose elements the symbol $j_p^1 f$ is used. We shall call each one of these classes "1-jets" of *f*. Next we have the:

**Definition 1.** The first jet bundle of $S(B)$ is the manifold:

$$J^1(B) = \{j_p^1 f | p \in M, f \in S(B)\}$$

We take the local representation: $J^0(B) = M \times F$ where, as usual, $F_p = \pi^{-1}(p)$ is the fiber over $p \in M$, such that at each point there is an isomorphism of each $F_p$ to *F*. Higher order jet bundles are defined likewise. All this stuff is abstract however we shall work with local representatives. For that reason, we introduce on each local chart of $J^1(B)$ an adapted coordinate system of the form:

$$\left\langle x_1, \ldots, x_n; A^1, \ldots, A^m; A_1^1, \ldots, A_n^1, \ldots, A_1^m, \ldots, A_n^m \right\rangle = def \left\langle x_i; A^j; A_i^j \right\rangle$$

So locally we can write $J^1(B) = M \times F \times K$ with $\dim M = n$, $\dim F = m$, $\dim K = nm$. Of course, a local coordinate system for $J^0(B)$ can be written as $\left\langle x_i; A^j \right\rangle$. For a given local section: $j^1 f : M \to J^1(B)$ we shall have:

$$\langle x_i;\ f^j(x_i);\ \frac{\partial f^j}{\partial x_i}(x_i)\rangle,\ i = 1, ..., n. \quad j = 1, ..., m.$$

Here we have already introduced the shorthand notation we will use throughout the article. So we can see that the section $f$ annihilates the contact 1-forms:

$$g_j = dA^j - A_i^j dx_i$$

i.e. $f^* g_j = 0$. A partial differential equation of the first order is in this framework, an immersed submanifold, but we are more interested in lagrangian functions.

**2.2. The equivalence problem**

There are many approaches to discussing symmetries of differential equations but the lagrangian method and its Hamiltonian counterpart are by far the most popular in Physics and sometimes are much better adapted than Lie methods, that deals directly with the differential equations, to treat matters of symmetry because while using Lie methods, it is possible to find symmetries accepted by the differential equations that are not symmetries of the lagrangian functional (see, on the subject of Lie symmetries e.g. [8]), we can be safe that all symmetries accepted by the lagrangian functional are accepted by the Euler-Lagrange equations of the problem, simplifying its handling because we can work directly with the lagrangian functional and not with the differential equations themselves. Besides using Noether Theorem, it is possible to relate symmetries to conservation laws. Therefore to discuss symmetries the lagrangian approach can be invaluable, as we shall see. So let us suppose that we have a lagrangian $L : J^1(B) \to R$, with $R$ the real number system and a lagrangian functional:

$$l(f) = \int_V j^1 f^* L(x_i;\ A^j;\ A_i^j) dV \tag{1}$$

Where $V$ is a compact subset of $M$, so we are leaving aside radiation problems and $dV$ is the volume element of $M$ which we can write down in local coordinates as: $dV = dx_1 dx_2, .., dx_n$. The variational problem is always around a local section of the 1-jet bundle, so we have written $j^1 f^* L$ to point out this fact, but we are more interested in questions about the invariance properties of the lagrangian functional, so we have to deal with local maps $\rho : J^1(B) \to J^1(B)$ which are given by:

$$\overline{x}_i = \overline{x}_i(x_i;\ A^j;\ A_i^j),\ \overline{A}^j = \overline{A}^j(x_i;\ A^j;\ A_i^j),\ \overline{A}_i^j = \overline{A}_i^j(x_i;\ A^j;\ A_i^j) \quad (2)$$

We can see (2) as a member of a family of transformations that are close to a Lie group, but that is not really important at this point. For our purposes, a neat statement of the equivalence problem has been given by [11]:

"Equivalence Problem: Given two Lagrangian functions $L[u]$ and $\overline{L}[\overline{u}]$ when does there exist a change of variables.. (2).. taking $L$ to $\overline{L}$ ?. If so, how does one explicitly construct the change of variables"?

In this citation we have inserted the explicit reference to our transformation (2). We pretend to solve a particular equivalence problem in this article for the case of Maxwell's equations gauge transformations but we shall not use Olver methods but a direct approach. To advance we need to define the behavior of the lagrangian under sets of transformations. The concept which we shall find useful is the concept of *s*-equivalence.

**Definition 2.** Given two lagrangians $L$ and $\overline{L}$ we say that they are *s*-equivalent if and only if, there exists a *n*-vector function $F : J^l(B) \to R^n$, with an arbitrary entire number, such that $L = \overline{L} + divF$ where *div* is the divergence operator in terms of the total derivative operators $D_i = \dfrac{\partial}{\partial x_i} + A_i^j \dfrac{\partial}{\partial Aj} + \ldots$

Most commonly the vector function $F$ is supposed with a domain in $R^n$, however this doesn't seems to be a necessary condition. Obviously when so defined the total derivative operators reduce to ordinary partial derivatives. Olver [11] did not use a name for *s*-equivalence but calls a divergence a "null lagrangian", however [12] works with def. 2. A slightly different sense is used by [13]. We shall not go into a critical discussion of the $s$-equivalent concept. For us def. 2 is enough. See, however [14] chap. V section 2.

The importance of $s$-equivalence relies in the fact that two $s$-equivalent lagrangians have the same Euler-Lagrange (EL) equations. So if a transformation like (2) amounts to $s$-equivalence of lagrangians there is no change in the variational problem and the transformation is a symmetry of the EL equations. If lagrangians are not $s$-equivalents the EL equations change in form, showing that the transformation is not one of its symmetries. When dealing with transformations like (2) the lagrangian is the horizontal $n$-form: $j^1 f^*(LdV)$ however in this paper such generality is not required because we shall not consider transformations of the base manifold coordinates, therefore we shall work with the lagrangian functions only. In definition (3) the lagrangians are arbitrary functions, but what we need is invariance, i.e., we need that both lagrangians share the same functional form. Hence let us introduce the obvious:

**Definition 3.** We say that a lagrangian $L$ is $s$-equivariant if and only if, under the action of the group representation $H : G \times J^1(B) \to J^1(B)$ of the Lie group $G$ it preserves its functional form up to a divergence; i.e. $H_g^* L = L + divF$.

This definition will be our working definition. Now we must introduce the kind of transformations we are interested in.

**Definition 4.** When the transformations involve arbitrary functions we call them "gauge transformations".

Gauge transformations are not uncommon in pure mathematics, indeed many partial differential equations accept gauge symmetries; most notably inhomogeneous linear partial differential equations. But many others accept these symmetries too. The important set of explicit gauge transformations for us is the one given by:

$$\bar{x}_i = x_i,\ \overline{A}^j = \overline{A}^j(x_i;\ A^j;\ A_i^j;\ F_i^k),\ \overline{A}_i^j = \overline{A}_i^j(x_i;\ A^j;\ A_i^j;\ F^k,\ F_i^k) \quad (2a)$$

Usually referred to as “internal transformations” because they only transform the $A$ coordinates. In the transformation (2a) we have explicitly introduced a set of arbitrary functions and its derivatives, but we must be clear that, using def. 4, we can say that when a lagrangian admits any sort of transformations e.g. (2) the EL equations are not only generally covariant, but gauge invariant too.

### 2.3. The Maxwell’s equations

Let us now take $n = m = 4$ and introduce Gaussian units for the Maxwell’s equations. It is well-known that Maxwell’s equations without taking into account constitutive relations are generally covariant for transformations of the sort: $\bar{x}_i = \bar{x}_i(x_i)$, included in (2) (see e.g. [14, 15 and 16]. But when the Lorentz ether relations are taken into account the symmetry group of space-time diffeomorphisms is broken, leading to something like the Lorentz symmetry group (on constitutive relations see e.g. [16]. So the gauge symmetries of the Maxwell’s equations arising from covariance under space-time diffeomorphisms are lost, but there is yet another transformation of the general form (2a) that is a gauge symmetry not broken when we choose a set of constitutive relations. This transformation looks very innocent and quite simple, but it implies a lot of consequences in many fields related to electromagnetic theory. Because we want to work in the vacuum we suppose that the Lorentz ether constitutive relations are applied, therefore the polarization and magnetization vectors are zero. That said the transformations we are interested in are given by:

$$\overline{A}^j A^j + \partial_j F, \; j = 1, \dots, 3, \; \overline{A}^4 = A^4 - \frac{1}{c}\partial_4 F \tag{3}$$

We take the 4 coordinate as time. In these transformations $F : R^4 \to R$ is an arbitrary function. We are not going to consider any combinations of (2) with (3) see, e.g., [16]. It is routine to prove that (3) are indeed a group but an infinite dimensional one which we call $G_g$ for short. This group is a group of contact transformations because an easy calculation shows that transformations (3) formally preserve each contact form $g_i$. The equivalence problem becomes now the problem of determining the group, $G_g$, i.e. a class of functions $F$ for a given lagrangian. This is the final statement of the problem we propose to solve: given a lagrangian, how does one explicitly construct the gauge transformations (3) that leave it $s$-equivariant?. These transformations are symmetries of the EL equations. This is a very general formulation still so at the end of section III.1 we shall give a more refined statement of our problem.

Transformations (3) are not the most general gauge transformations. In the theory of gauge fields, as introduced by Yang and Mills, a gauge field is required to obtain the invariance of the lagrangian in front of so-called "local gauge transformations" (see e.g. [3] or [15] chap. 10). Hence if we have $N$ fields $\varphi^m$ with transformation law

$$\overline{\varphi}^m = \sum_{k=1}^{N} g_{mk}(x_i)\varphi^k \tag{3a}$$

The lagrangian will be gauge invariant if and only if gauge fields $A^j$ are introduced with a specific transformation law of which (3) is a special case. This general transformation law will not be of our concern here, hence we limit the concern of this paper to (3), which we shall call "additive gauge transformation".

Indeed if we start from the Maxwell's lagrangian the problem is quite trivial because the lagrangian is fully invariant in front of (3), hence (3) is a set of gauge symmetries of the EL equations. However, just like when we use constitutive relations to break up the general group of diffeomorphisms there is a widely-used procedure to break the invariance of Maxwell's equations in front of (3). Such procedure is known as "gauge fixing". As we shall see when we choose a gauge we can solve the equivalence problem by specifying a class of functions $F$ that define a group of gauge symmetries for our lagrangian, leaving invariant the EL equations.

## 3. Choosing a Gauge as a Sieve for Gauge Transformations

### 3.1. The lagrangians

Let us put forth the Maxwell's lagrangian using the potentials only; not the field strengths:

$$L_m = \sum_i^3 \left(\frac{\partial A^4}{\partial x}\right)^2 - \frac{2}{c}\sum_i^3 \frac{\partial A^4}{x_i}\frac{\partial A^i}{\partial x_4} + \frac{1}{c^2}\sum_i^3 \left(\frac{\partial A^i}{\partial x_4}\right)^2$$

$$- \sum_i^3 \left(\sum_{jk}^3 \in_{ijk} \frac{\partial A^j}{\partial x_k}\right)^2 - 4\pi\rho A^4 + \frac{4\pi}{c}\sum_i^3 A^j J_i \qquad (4)$$

We have refrained from using Gibbs 3-dimensional notation to write down this lagrangian because we want to show the local section on the fiber coordinates, which in the lagrangian (4) has been inserted already using, as an abuse of notation, the A's instead of the *f* used in section II.1; the symbol $\in_{ijk}$ means the Levi-Civita pseudotensor, the indexes run from 1 to 3 unless otherwise stated. To simplify even more we shall put: $A^4 =_{def} A$ and $x_4 = de_f\ t$ and instead of subscript 4 we would rather use subscript when we refer to a time partial derivative. With this notation the lagrangian functional (1) is known as an action functional. A straightforward direct calculation shows that (4) is exactly invariant under transformations (3) when the matter fields $\rho, J_i$ are not considered and up to a divergence when they are included and satisfy the continuity equation. Hence it is clear that charge conservation is a constraint deduced not only from field equations, but more fundamentally from gauge invariance, as is well known. Our immediate interest is on the procedures of symmetry breaking induced by choosing a gauge. For that purpose we introduce the following definition:

**Definition 5.** A gauge is a submanifold of $J^1(B)$.

Because of the way we defined $J^1(B)$ any one of its submanifolds is sectionable. Clearly this is not the definition one learns form electromagnetic theory textbooks. Hence to break the general gauge symmetry of Maxwell's equations we must include the general lagrangian in one of the submanifolds by the gauge defined. Now let us introduce the most important gauges we shall deal with: the Coulomb and Lorenz gauges.

**Definition 6.** The Coulomb gauge is a submanifold of $J^1(B)$ defined by:

$$S_C = \{p \in J^1(B) \Big| \sum_{i=1}^{3} A_i^i = 0\}$$

**Definition 7.** The Lorenz gauge is a submanifold of $J^1(B)$ defined by:

$$S_L = \left\{p \in J^1(M) \Big| \sum_{i=1}^{3} A_i^i + \frac{1}{C} A_t = 0\right\}$$

Now we know that the Coulomb gauge lagrangian is a function such that: $L_c : S_c \to R$, while the Lorenz gauge lagrangian is such that: $L_L : S_L \to R$. There are two obvious paths to write down the Maxwell's lagrangian on each of the manifolds defined by the gauges, the most simple, we believe, is to find the way to introduce the condition within the Maxwell's lagrangian using $s$-equivalent lagrangians where the condition can be inserted, i.e., we use a map to the corresponding submanifold i.e.. something like $u : J^1(B) \to S_C$. The other path involves the use of Lagrange multipliers which are useful to make the calculations even more laborious. In both cases what is involved is the inclusion map, but in the first procedure it is explicitly defined. In this paper we shall follow the first path only because with quite simple transformations we can find the right lagrangians. Let us show how.

There are three terms in the Maxwell's lagrangian that do not change under additive gauge transformations: the magnetic term and the coupling terms while the continuity equation remains valid. Hence we must look at the first three terms only. As a matter of fact the most important terms are those that define the electric field strength.

Let us write the lagrangian (4) as:

$$L_M = \sum_i^3 \left(\frac{\partial A}{\partial x_i}\right)^2 + \frac{1}{c^2} \sum_i^3 \left(\frac{\partial A^i}{\partial t}\right)^2 - \sum_i^3 \left(\sum_{jk}^3 \in_{ijk} \frac{\partial A^j}{\partial x_k}\right)^2$$

$$- 4\pi\rho A + \frac{4\pi}{c} \sum_i^3 A^i J_i + \frac{2}{c} A \frac{\partial}{\partial t}\left(\sum_{i=1}^3 \frac{\partial A^i}{\partial x_i}\right) \qquad (5)$$

In this lagrangian we have dropped the divergence terms. This new lagrangian is $s$-equivalent to Maxwell's lagrangian, hence nothing new is involved. But if we look at the last term we can see that the inclusion in the set $S_C$ is now an easy task that gives us:

$$L_C = \sum_i^3 \left(\frac{\partial A_C}{\partial x_i}\right)^2 + \frac{1}{c^2} \sum_i^3 \left(\frac{\partial A_C^i}{\partial t}\right)^2$$

$$- \sum_i^3 \left(\sum_{jk}^3 \in_{ijk} \frac{\partial A_c^J}{\partial x_k}\right)^2 - 4\pi\rho A_C + \frac{4\pi}{c} \sum_i^3 A_C^i J_i \qquad (6)$$

In this new lagrangian, which we call "Coulomb lagrangian", the gauge has been taken into account so we have included our original Maxwell's lagrangian on an open subset of the submanifold $S_C$ as desired. Following the same path we can arrive at the "Lorenz lagrangian", which is defined on an open subset of $S_L$ and is of the explicit form:

$$L_L = \sum_i^3 \left(\frac{\partial A_L}{\partial x_i}\right)^2 + \frac{1}{c^2} \sum_i^3 \left(\frac{\partial A_L^i}{\partial t}\right)^2 - \frac{2}{c^2} A_L \frac{\partial^2 AL}{\partial t^2}$$

$$- \sum_i^3 \left(\sum_{jk}^3 \in_{ijk} \frac{\partial A_L^j}{\partial x_k}\right) - 4\pi\rho A_L + \frac{4\pi}{c} \sum_i^3 A_L^i J_i \qquad (7)$$

The Coulomb and Lorenz lagrangians are not generally gauge invariant, hence we can state more precisely our problem: "Given the Lorenz and Coulomb lagrangians, how does one construct its groups of gauge symmetries if they exist"? Another question that we can solve is if there exists a gauge transformation $s$-relating the Lorenz and Coulomb lagrangians. These are our tasks in the next section.

### 3.2. Gauge symmetries for Coulomb and Lorenz lagrangians

The procedure is going to be direct, we insert the transformations (3) on each lagrangian to immediately see that they are not $s$-equivariant for a general additive group of gauge transformations, hence we impose some obvious constraints on the gauge function $F$ that produce $s$-equivariance. These constraints give us the class, with trasfinite cardinality, from which function $F$ must be taken to maintain $s$-equivariance. Therefore we obtain the class of gauge symmetries of our lagrangian and of our EL equations. We are going to enunciate our results as theorems for facilitating discussion. In all that follows the functions involved are of class $C^N(R^n, R^m)$ with $N$ as large as desirable with the adequate integers $n$, $m$ and all derivatives at all orders commute.

We start with a pair of easy lemmas:

**Lemma 1.** *Under the action of the gauge group* $G_g$ *the Coulomb lagrangian* $L_C : S_c \to R$ *is given by*:

$$H_F{}^* L_C = L_C(\overline{A}_C, \overline{A}^i_C, \partial_i \overline{A}_C, \partial_t \overline{A}^i_C) + \frac{2}{c} \sum_{i=1}^3 \partial_i \overline{A}_C \partial_i \partial_t F$$

$$- \frac{2}{c^2} \sum_{i=1}^3 \partial_t \overline{A}^i_C \partial_t \partial_i F + \frac{1}{C^2} \sum_{i=1}^3 (\partial_t \partial_i F)^2 \qquad (8)$$

**Proof.** It is a straightforward direct substitution of the gauge transformations (3) in the Coulomb lagrangian (6) making use of the Coulomb condition where needed. QED

Using well-known identities we can find the following $s$-equivalent lagrangian to (8):

$$H_F{}^* L_C = L_C(\overline{A}_C, \overline{A}^i_C, \partial_i \overline{A}_C, \partial_t \overline{A}^i_C) + \frac{2}{c} \partial_t F \Delta \overline{A}_C$$

$$- \frac{1}{c^2} \partial_t F \partial_t \sum_{i=1}^3 \partial_i \overline{A}^i_C + \frac{1}{c^2} \partial_t F \Delta \partial_t F + \mathrm{div}\mathbf{k} \qquad (9)$$

With the 3-vector function $\boldsymbol{K}$ is given by its Cartesian components:

$$\mathrm{K_i}\ \frac{1}{C^2}\frac{\partial F}{\partial t}\frac{\partial A^i}{\partial t}+\frac{1}{c}\frac{\partial F}{\partial t}\frac{\partial}{\partial x_i}\frac{\partial F}{\partial t}-\frac{1}{c}\overline{A}\frac{\partial}{\partial x_i}\frac{\partial F}{\partial t} \tag{10}$$

We need one more lemma related to the invariance of the gauge submanifolds under the action of the gauge groups, otherwise if the group action does not leave invariant the gauge submanifolds we can go away from it along a group orbit, hence along this orbit the corresponding field equations are not satisfied, so the group is not a symmetry group:

**Lemma 2.** *The Coulomb gauge manifold* $S_c$ *is invariant under the gauge group action if and only if*, $F$ *is a time dependent solution to Laplace equation*.

**Proof.** When we apply a general element of the gauge group $G_g$ to the gauge manifold we obtain:

$$\sum_{i=1}^{3}\partial_i A_C{}^i = \sum_{i=1}^{3}\partial_i \overline{A}_C{}^i + \Delta F \tag{11}$$

Now if the function $F$ is a time dependent solution of Laplace equation the Coulomb gauge is invariant and if the Coulomb gauge is invariant (i.e. $\sum_{i=1}^{3}\partial_i A_C{}^i = \sum_{i=1}^{3}\partial_i \overline{A}_C{}^i$), $F$ must be a time dependent solution of Laplace equation as can be seen from (11). QED

**Theorem 1.** *The Coulomb lagrangian is s-equivariant for two non-zero solutions* $\overline{A}_C^j$, $\overline{A}_C$, $A_C^j$, $A_C$ *of the field equations if and only if*, *its gauge group is given by*:

$$\overline{A}_C{}^j = A_C{}^j + \frac{\partial F_c}{\partial x_j},\ \overline{A}_c = A_c \tag{12}$$

Where $F_c : R^3 \to R$ is any time independent solution of Laplace equation? We shall call this gauge group the "Coulomb Gauge Group" $G_c$

**Proof.** We can see that any element of the Coulomb group leaves invariant the Coulomb lagrangian, hence, the conditions are sufficient for *s*-equivariance. Now, if any element of the gauge group acts on the

Coulomb manifold the Coulomb lagrangian is given by lemma 1, so, if it is $s$-equivariant, the function $F$ must satisfy:

$$\frac{2}{c}\partial_t F \Delta \overline{A}_C - \frac{1}{c^2}\partial_t F \partial_t \sum_{i=1}^{3} \partial_i \overline{A}_C^i + \frac{1}{c^2}\partial_t F \Delta \partial_t F = 0 \qquad (13)$$

Because of the definition of Coulomb manifold the second term is zero and because of lemma 2 the third term is zero too (we suppose $F$ is as smooth as desired). Hence just the first term survives. But the laplacian of $\overline{A}_C$ cannot be arbitrarily defined because it is given by the field equations, therefore the time derivative of $F$ must be zero. QED

Hence from Theorem (1) is clear that $K_i = 0$ for all $i$, so there are not Noether charges conserved under the action of the Coulomb gauge group $G_C$. And, probably more important, there are not surface terms to be considered when the boundary value problems are considered. This is a feature of the Coulomb gauge.

Now we shall make the same treatment for the Lorenz gauge lagrangian.

**Lemma 3.** *Under the action of the gauge group* $G_g$ *the Lorenz lagrangian* $L_L : S_L \to R$ *is given by*:

$$H_{F_0}{}^{*}L_L = L_L(\overline{A}_L, \overline{A}_L^i, \partial_i \overline{A}_L, \partial_t \overline{A}_L^i) + \frac{2}{c}\overline{A}_L \partial_t F_0 - \frac{2}{c^2}\partial_t F_0 \partial_t F_0 + \mathrm{div}\mathbf{P} \qquad (14)$$

We have used $= \Delta - \frac{1}{c^2}\partial_t$ for the D'Alembert operator. The 3-vector function $\boldsymbol{P}$ is given by its Cartesian components:

$$P_i = \frac{2}{c^2}\partial_t F_0 \partial_t \overline{A}^i + \frac{1}{c^2}\partial_t F_0 \partial_i \partial_t F_0 - \frac{2}{c}\partial_t F_0 \partial_i \overline{A} \qquad (15)$$

**Proof.** It is a straightforward direct substitution of the gauge transformations (3) in the Lorenz lagrangian (7) making use of the Lorenz condition where needed. QED

**Lemma 4.** *The Lorenz gauge manifold* $S_L$ *is invariant under the gauge group action if and only if, the function* $F_0$ *is solution to D'Alembert equation.*

**Proof.** Again it is straightforward. We apply a general element of the gauge group $G_g$ to the Lorenz gauge manifold condition

$$\sum_{i=1}^{3} \partial_i A_L^i + \frac{1}{c}\partial_t A_L = \sum_{i=1}^{3} \partial_i \overline{A}_L^i + \frac{1}{c}\partial_t \overline{A}_L + F_0 \tag{16}$$

So the condition is sufficient, because if $F_0$ is a solution to D'Alembert equation the Lorenz gauge manifold is invariant and if the Lorenz gauge manifold is invariant, then $F_0$ is a solution to D'Alembert equation. QED

**Theorem 2.** *The Lorenz lagrangian is s-equivariant for two non-zero solutions* $\overline{A}_L^j$, $\overline{A}_L$, $A_L^j$, $A_L$ *of the field equations if and only if, its gauge group is given by*:

$$\overline{A}_L{}^j = A_L{}^j + \frac{\partial F_0}{\partial x_j},\ \overline{A}_L = A_L - \frac{1}{c}\frac{\partial F_0}{\partial t} \tag{17}$$

Where $F_0 : R^4 \to R$ is any solution to D'Alembert equation? We shall call this gauge group the "Lorenz Gauge Group" $G_L$

**Proof.** The condition is clearly sufficient using lemma 3. From lemma 4 we know that $F_0$ must be a solution to D'Alembert equation, so it is a necessary condition for $s$-equivariance. But, again using lemma 3 we can see that if the Lorenz lagrangian is $s$-equivariant we must have

$$\frac{2}{c}\overline{A}_L \partial_t F_0 - \frac{2}{c^2}\partial_t F_0 \partial_t F_0 = \frac{2}{c}\left(\overline{A}_L - \frac{1}{c}\partial_t F_0\right)\partial_t F_0 = 0$$

The first factor cannot be zero by hypothesis, so $F_0$ must be a solution to D'Alembert equation. QED

For the Lorenz gauge there is an explicit non zero conserved Noether current with components $P_i$.

We can see that the "Coulomb Gauge Group" leaves invariant the scalar potential in the Coulomb gauge, while this is not the case for the scalar potential of the Lorenz gauge. This is a most unexpected result, showing that if the electric and magnetic fields are "real" fields because they are gauge invariant, hence the scalar potential in the Coulomb gauge must share this ontological status being "real" as well. If this is the case there must be physical effects. But if this is correct and the Coulomb scalar potential is a "real" physical quantity and the gauge invariant criterion of "reality" is right, on these philosophical grounds we can assert that there is not any additive gauge transformation relating a "real" quantity the gauge invariant Coulomb potential-with a fictitious one like the Lorenz non-gauge invariant scalar potential. Now we must prove that this is the case within the mathematical framework that we have introduced all along this article. As previously we start with a few lemmas.

**Lemma 5.** *There exists an invertible additive gauge transformation* (3) *such that*

$$\mu : S_c \to S_L,$$

*given explicitly by*:

$$A_C^i = A_L^i + \partial_i F,\ A_C = A_L - \frac{1}{c}\partial_t F, \tag{18}$$

*if and only if the gauge function F satisfies*

$$\Delta F = \frac{1}{c}\partial_t A_L \ldots\ F = \frac{1}{c}\partial_t A_C \tag{19}$$

**Proof.** It is a straightforward calculation using the conditions defining the Coulomb and Lorenz gauge manifolds. So let's suppose that $F$ exists, therefore we can substitute (18) in the Lorenz gauge condition to get:

$$\sum_{i=1}^{3} \partial_i A_L^i + \frac{1}{c}\partial_t A_L = \sum_{i=1}^{3} \partial_i A_C^i - F + \frac{1}{c}\partial_t A_C \tag{20}$$

And in the Coulomb gauge condition to obtain:

$$\sum_{i=1}^{3} \partial_i A_C^i = \sum_{i=1}^{3} \partial_i A_L^i + \frac{1}{c} \Delta F \tag{21}$$

Hence if $F$ exists and relates both gauges when the transformation acts on the manifolds, it must transform one manifold into another, this can be done if (19) are satisfied as can be seen by direct substitution of (19) in (20) and (21). Hence (19) is sufficient. Now, if the condition is necessary its violation implies that no gauge transformation exists between the manifolds. If we see at (20) and (21) we discover that if the conditions are not satisfied the gauge transformation transforms the Coulomb (or Lorenz) manifold into another one. So (19) is necessary. QED

Conditions (19) are formal conditions on the gauge function $F$, a function that must be able to take us from the Lorenz gauge manifold to the Coulomb gauge manifold and conversely. Jackson deduced the condition for the transformation from the Lorenz to the Coulomb gauge (see [2] Eq. 3.8) using explicit solutions to the differential equations, but he did not deduce the conditions for the inverse transformation. These conditions define a gauge group of transformations which we shall call $G_{LC}$ for short. But these conditions are not the whole set of conditions: a few more are added when we impose the condition that the Lorenz lagrangian is *s*-equivalent to the Coulomb lagrangian through the same gauge function $F$.

**Lemma 6.** *Under the action of the group* $G_{LC}$ *the Coulomb lagrangian becomes* (*the magnetic term is clearly gauge invariant*, *while if we want charge conservation the coupling terms are invariant too*, *so we can omit them from the analysis*):

$$H_F^* L_C = \sum_i^3 \left( \frac{\partial A_L}{\partial x_i} \right)^2 + \frac{1}{c^2} \sum_i^3 \left( \frac{\partial A_L^i}{\partial t} \right)^2 - \frac{2}{c} \sum_{i=1}^{3} \partial_i A_L \partial_i \partial_t F$$

$$+ \frac{2}{c^2} \sum_{i=1}^{3} (\partial_i \partial_t F)^2 + \frac{2}{c^2} \sum_{i=1}^{3} \partial_t A_L^i \partial_i \partial_i F + \dots \tag{22}$$

**Proof.** Straightforward direct substitution of (18) in the Coulomb lagrangian. QED

Up to a divergence we can write down (22) as follows:

$$H_F^* L_C = \sum_i^3 \left(\frac{\partial A_L}{\partial x_i}\right)^2 + \frac{1}{c^2}\sum_i^3 \left(\frac{\partial A_L^i}{\partial t}\right)^2$$

$$+\frac{2}{c}\left(A_L - \frac{1}{c}\frac{\partial F}{\partial t}\right)\Delta\partial_t F + \frac{2}{c^3}\,\partial_t F \partial_t^2 A_L + \ldots + \mathrm{div}\mathbf{O} \tag{23}$$

Where the 3-vector $\boldsymbol{O}$ has Cartesian components:

$$O_i = -\frac{2}{c}A_L\partial_i\partial_t F + \frac{2}{c^2}\partial_t F\partial_i\partial_t F + \frac{2}{c^2}\partial_t F + \frac{2}{c^2}\partial_t F\partial_t A_L^i \tag{24}$$

**Theorem 3.** *There is no gauge function F, for non-zero non-trivial (trivial means* $A_C = A_L$*) solutions in each gauge, such that the Coulomb lagrangian becomes s-equivalent to the Lorenz lagrangian.*

**Proof.** Directly inspecting (23) we can see that if such a function exists we must have:

$$\Delta\partial_t F = 0,\ \partial_t F = -A_L \tag{25}$$

hence:

$$\Delta A_L = 0 \tag{26}$$

However, conditions (25) are in open contradiction to lemma 5 and condition (26) is contradictory with the field equations of the Lorenz gauge. Hence no such a function exists. QED

**Lemma 7.** *Under the action of the group* $G_{LC}$ *the Lorenz lagrangian becomes (same caveat that in lemma* 6*):*

$$H_F^* L_L = \sum_i^3 \left(\frac{\partial A_C}{\partial x_i}\right)^2 + \frac{1}{C^2}\sum_i^3 \left(\frac{\partial A_C^i}{\partial t}\right)^2 - \frac{2}{C^3}A_C\partial_t^2 A_C$$

$$-\frac{2}{C}\sum_{i=1}^3 \partial_i A_C \partial_i\partial_t F + \frac{2}{C^2}\sum_{i=1}^3 (\partial_i\partial_t F)^2$$

$$+\frac{2}{C^2}\sum_{i=1}^{3}\partial_t A_C^i \partial_i \partial_i F+\frac{2}{C^4}A_C \partial_t^3 F$$

$$+\frac{2}{C^4}\partial_t F \partial_t^2 A_c-\frac{2}{C^5}\partial_t F \partial_t^3 F \tag{27}$$

**Proof.** Just like before: a straightforward direct substitution of (18) in the Lorenz lagrangian. QED

We can find an $s$-equivalent lagrangian to (27) given by:

$$H_F^* L_L = \sum_i^3 \left(\frac{\partial A_C}{\partial x_i}\right)^2 + \frac{1}{c^2}\sum_i^3 \left(\frac{\partial A_C^i}{\partial t}\right)^2 + \frac{2}{c}\left(A_C - \frac{1}{c}\partial_t F\right)\Delta \partial_t F$$

$$-\frac{2}{C^2}\left(A_C - \frac{1}{c}\partial_t F\right)\partial_t^2 A_C + \frac{2}{c}\left(A_C - \frac{1}{c}\partial_t F\right)\partial_t^3 F \tag{28}$$

The Noether current is quite the same as (24). To get (28) the Coulomb gauge condition is used.

**Theorem 4.** *There is no gauge function F for non-zero non-trivial solutions, such that the Lorenz lagrangian becomes s-equivalent to the Coulomb lagrangian.*

**Proof.** In the expression (28) we factorize $\frac{2}{c}\left(A_C - \frac{1}{c}\partial_t F\right)$, a term that cannot be zero, to obtain the condition of $s$-equivalence:

$$\Delta \partial_t^* F + \frac{1}{c^2}\partial_t^3 F - \frac{1}{c}\partial_t^2 A_C = 0 \tag{29}$$

a condition in conflict with lemma 5. Let us elaborate on this point. We can add the time derivative of the equation $F = \frac{1}{c}\partial_t A_C$ to (29) to obtain:

$$\Delta \partial_t F = \frac{2}{c}\partial_t^2 A_C \tag{30}$$

Hence we can write down taking into account lemma 5: $\partial_t^2(2A_C - A_L) = 0$. If we remember that: $A_C - A_L = -\frac{1}{c}\partial_t F$ we can get: $\partial_t^2 A_C + \partial_t^2(A_C - A_L) = 0$ in the form:

$$\frac{1}{c^2}\partial_t^3 F = \frac{1}{c}\partial_t^2 A_C \tag{31}$$

Now we insert equation (31) in (29) to obtain: $\Delta\partial_t F = 0$ which is in obvious conflict with the equation: $\Delta\partial_t F = \frac{1}{c}\partial_t^2 A_L$ except in case: $\partial_t^2 A_L = 0$, but in that case there is nothing to prove, because the field equations of the Lorenz gauge coincide with those of the Coulomb gauge. QED

In this way we have showed that there is not an additive gauge transformation relating the Coulomb gauge manifold and the Lorenz gauge manifold. Taking a close look at the proofs we can identify the reason: the gauge function $F$ is clearly overdetermined by the set of conditions that arise from (19) and the $s$-equivalence of the lagrangians involved. But Maxwell's equations are lagrangian equations and its symmetries can be treated at this level of abstraction. A clue to the result must have been seen in the fact that an elliptic partial differential equation like the equation for the scalar potential in the Coulomb gauge cannot be equivalent to a hyperbolic partial differential equation the equation for the scalar potential in the Lorenz gauge as was pointed out by [18 and 19] in his treatment of the question. Another clue was the proof that there exist potentials invariants in front of gauge transformations, where a prominent role was played by Helmholtz Theorem [5]. However, as can be easily showed, from the point of view of the lagrangian formalism the lagrangian obtained using the Helmholtz Theorem is $s$-equivalent to the Coulomb lagrangian as was showed directly by [20] for that reasons no new things can arise from its use, except the proof of gauge invariance of the potentials. Here we have obtained the exact group of gauge transformations that leave invariant the Coulomb lagrangian, a group that cannot give rise to any overdetermination of the boundary value problem for the potentials because the scalar potential is invariant, while the arbitrary function added to the vector potential is a harmless time independent solution of the Laplace equation; which is like adding a solution of the homogeneous problem: i.e. we obtained a subset of the general solution of the inhomogeneous D'Alembert equation. However, as we showed in [5] using the Helmholtz Theorem we can eliminate even this gauge symmetry using the solenoidal component of the vector potential. Hence we have proved that in the Coulomb gauge we can leave aside all gauge symmetries from the differential equations.

## Conclusions

If our result is quite correct and can be extended to any gauge, it does not mean that some empirically well-established results e.g. special relativity are flawed. It means that some suppositions related to mathematical matters were not seriously considered in the past. An analogy that could be useful is as follows: no one believes that any solution to Einstein equations is a physically relevant model of the actual universe, however it does not mean that Einstein equations are flawed or that our universe is the Godel universe. We think that our result shows that Maxwell equations with a gauge included represents different mathematical models of the world that can be used in different situations. In a situation where the velocity of the particles or the field propagation is at play the Lorenz gauge is the correct choice. So radiation theory is fully worked in the Lorenz gauge. However when the phenomena are static or quasistatic the Coulomb gauge could be the right choice. What Maxwell equations cannot give us is a unitary representation of the world able to accommodate all our prejudgments.

In this paper we have achieved the following results:

(*i*) We have calculated the explicit gauge groups for the Lorenz and Coulomb gauges solving an equivalence problem for each of them.

(*ii*) We have proved that there is not any gauge transformation relating the Coulomb and the Lorenz gauges.

(*iii*) We made clear the role of the Helmholtz Theorem in classical electrodynamics: it provides a way to eliminate the gauge symmetry of the Coulomb gauge.

Therefore we have extended and refined the results of [5]. Hence Onoochin and Engelhardt are right: the Helmholtz Theorem is not different from the Coulomb gauge, its role is to provide a way to show how to leave aside gauge symmetries. So, indeed, we have proved that in the Coulomb gauge we can obtain gauge invariant potentials, which is reason enough to review our criteria of "reality", because the potentials in the Coulomb gauge are as real as the field strengths according to generally accepted ideas.

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