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No-go theorem for the classical Maxwell-Lorentz electrodynamics in odd-dimensional worlds

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Abstract – If the conventional Maxwell-Lorentz formulation of classical electrodynamics is adopted in a flat spacetime of arbitrary odd dimension, then the retarded vector potential A^μ generated by a point charge turns out to be pure gauge, $A^\mu = \partial^\mu \chi$. By Gauss' law, the charge shows up as zero. The classical electromagnetic coupling is thus missing from odd-dimensional worlds. If the action is augmented by the addition of the Chern-Simons term, then the classical interaction picture in the three-dimensional world becomes nontrivial.

Introduction. – In this note we present a surprising result that the classical electromagnetic interaction, realized as the Maxwell-Lorentz theory, is missing from odd-dimensional spacetimes. It is well known that the four-dimensional electrodynamics can be extended to any even dimensions to result in a consistent theory [1–3]. However, one can hardly conceive that a similar extension to odd-dimensional spacetimes gives rise to a classical picture in which any point charge generates zero field strengths, and hence, by Gauss' law, the electromagnetic coupling is effectively vanishing. On the other hand, nontrivial solutions to homogeneous Maxwell's equations are still available. Therefore, only free classical Maxwellian fields may exist in such worlds.

The odd-dimensional physics is of concern to us not only for methodological reasons but also in relation to the holographic principle (for a review see, *e.g.*, [4,5]), whose rationale often leads one to consider systems living in an odd-dimensional bulk, as, say, in the AdS_3/CFT_2 model, the popular setting for analysing the holographic correspondence. The no-go theorem discussed here may be of utility in such studies.

The second section gives a proof of this theorem. Notice that the no-go theorem is only valid for a genuine $(2n+1)$-dimensional realm, not for electromagnetic systems constrained in a $2n$-dimensional spacelike manifold which is actually immersed in a higher-dimensional spacetime. We take a closer look at this issue in the final section. In addition, we adduce an argument that the analytical form of interaction between the quantized Maxwell field and different charged fields is common to all dimensions. It transpires that if the action is augmented by the addition of the Chern-Simons term, then the three-dimensional classical picture becomes nontrivial.

The Maxwell-Lorentz theory in odd-dimensional spacetimes. – We begin with the conventional formulation of the Maxwell-Lorentz electrodynamics in a flat D-dimensional world. Let us set the metric $\eta_{\mu\nu} = \mathrm{diag}(1, -1, \ldots, -1)$, and adopt units in which the speed of light equals unity. We write the action

$$S = S_{\rm p} - \int \mathrm{d}^D x \left(j^\mu A_\mu + \frac{1}{4\,\Omega_{D-2}}\, F_{\mu\nu} F^{\mu\nu} \right), \qquad (1)$$

where $S_{\rm p}$ is the mechanical part of the action responsible for the particle behavior, Ω_{D-2} is the area of the $(D-2)$-dimensional unit sphere, $\Omega_{D-2} = 2\,\pi^{(D-1)/2}/\Gamma[(D-1)/2]$, and the field strength is expressed in terms of the vector potential, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. For $D = 3$, the action (1) should be augmented by the addition of the Chern-Simons term. However, we would like to compare even- and odd-dimensional cases within a unified framework, so that we ignore for a while any augmentations and employ the action (1), which is well suited to the even-dimensional electrodynamics [1–3].

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Varying A_μ in (1) gives Maxwell's equations:

$$\partial_\mu F^{\mu\nu} = \Omega_{D-2} j^\nu. \tag{2}$$

The linearity of Maxwell's equations makes it possible to restrict our consideration to the single-particle case. A particle with a δ-shaped distribution of the electric charge e, moving along an arbitrary world line $z^\mu(s)$, gives rise to the current

$$j^\mu(x) = e \int_{-\infty}^{\infty} \mathrm{d}s\, v^\mu(s)\, \delta^D(x - z(s)), \tag{3}$$

where $v^\mu = \mathrm{d}z^\mu/\mathrm{d}s = \dot{z}^\mu$ is the D-velocity of this particle.

If we impose the Lorenz condition on the vector potential to fix the gauge, $\partial_\mu A^\mu = 0$, then (2) becomes

$$\Box A^\mu = \Omega_{D-2} j^\mu. \tag{4}$$

The physically relevant solution to (4) is

$$A^\mu(x) = \int \mathrm{d}^D x\, G_{\mathrm{ret}}(x - x')\, j^\mu(x'), \tag{5}$$

where $G_{\mathrm{ret}}(x)$ is the retarded Green's function of the wave operator. Our main interest here is with odd-dimensions $D = 2n + 3$, $n = 0, 1, \ldots$. In this case (with reference to, *e.g.*, [2,6]), we have

$$G_{\mathrm{ret}}(x) = N_D^{-1}\, \theta(x_0) \left(\frac{\mathrm{d}}{\mathrm{d}x^2}\right)^n \frac{\theta(x^2)}{\sqrt{x^2}}, \quad N_D = \frac{\Gamma\left(n + \frac{1}{2}\right)}{\sqrt{\pi}}. \tag{6}$$

We insert (3) and (6) in (5). Denoting $R_\mu = x_\mu - z_\mu(s)$, where x_μ is the observation point, and $z_\mu(s)$ the emission point on the world line, we obtain

$$A^\mu(x) = \frac{e}{N_D} \int_{-\infty}^{\infty} \mathrm{d}s\, v^\mu(s)\, \theta(R_0) \left(\frac{\mathrm{d}}{\mathrm{d}R^2}\right)^n \frac{\theta(R^2)}{\sqrt{R^2}}. \tag{7}$$

Let us take the integration variable $\lambda = R^2$, and observe that $\mathrm{d}\lambda/\mathrm{d}s = -2R\cdot v$, to yield

$$A^\mu(x) = \frac{e}{2N_D} \int_0^{\infty} \mathrm{d}\lambda\, \frac{v^\mu}{R\cdot v} \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^n \frac{\theta(\lambda)}{\sqrt{\lambda}}. \tag{8}$$

For $n \geqslant 1$, the integral (8) diverges at $\lambda = 0$. This is due to the fact that the integrand of (5) is the product of two singular distributions. Since the retarded Green's functions are singular in themselves, they are normally integrated with smooth sources $j^\mu(x)$. To assign a mathematical sense to the ill-defined expression (8), we regularize it by the convention that the derivatives should act on the left and the surface term is ignored. According to this regularization prescription,

$$\begin{aligned} A^\mu(x) \;=\; & \frac{e(-1)^n}{2N_D} \int_0^{\infty} \frac{\mathrm{d}\lambda}{\sqrt{\lambda}} \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^n \frac{v^\mu}{R\cdot v} = \\ & \frac{e(-1)^n}{2N_D} \int_0^{\infty} \frac{\mathrm{d}\lambda}{\sqrt{\lambda}} \frac{\partial}{\partial x_\mu} \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^n \log(R\cdot v). \end{aligned} \tag{9}$$

With the estimations $R^\mu = O(\lambda^{1/2})$, $v^\mu = O(1)$ as $\lambda \to \infty$, valid for world lines that approach asymptotically to straight timelike lines, the integral (9) converges uniformly for $n \geqslant 1$, that is, for $D \geqslant 5$. Therefore, the order of integration and differentiation with respect to x^μ may be interchanged with the opposite one. As to $n = 0$, one may think of (9) as

$$A_\mu(x) = \frac{e}{2} \lim_{\epsilon\to 0} \frac{\partial}{\partial x^\mu} \int_0^{\infty} \frac{\mathrm{d}\lambda}{\sqrt{\lambda}}\, e^{-\epsilon\lambda} \log(R\cdot v). \tag{10}$$

Be it as it may, whenever the $\partial/\partial x_\mu$ and integration are commutative, we get $A^\mu = \partial^\mu \chi$.

In fact, the interchangeability of these operations is a subtle issue. The order of implementation of these operations must not be changed if the measure contains a singular component, such as the δ-function or its derivatives, because the integrals are no longer uniformly convergent. To illustrate this, we refer to $D = 4$. In lieu of (8), we have

$$\begin{aligned} A^\mu(x) \;=\; & e \int_{-\lambda_m}^{\infty} \mathrm{d}\lambda\, \delta(\lambda)\, \frac{v^\mu}{R\cdot v} = \\ & e \int_{-\lambda_m}^{\infty} \mathrm{d}\lambda\, \delta(\lambda)\, \frac{\partial}{\partial x^\mu} \log(R\cdot v), \end{aligned} \tag{11}$$

where λ_m is the maximal absolute value of the spacelike interval between a given point of observation x^μ and the world line. The ∂^μ must be applied prior to integration, and hence (11) results in $ev^\mu/(R\cdot v)|_{\mathrm{ret}}$, the Liénard-Wiechert vector potential, rather than $e\partial^\mu[\log(R\cdot v)]_{\mathrm{ret}}$ which differs from the Liénard-Wiechert vector potential in the term $eR^\mu[(R\cdot\dot{v}) - 1]/(R\cdot v)|_{\mathrm{ret}}$.

If we integrate (2) over a domain V containing the charged particle in the hyperplane perpendicular to the world line and take into account the relation $F_{\mu\nu} = 0$, we obtain

$$e\,\Omega_{D-2} = \oint_V \mathrm{d}^{D-1}x\; \partial_\mu F^{\mu\nu} v_\nu = \oint_{\partial V} \mathrm{d}^{D-2}x\; n_\mu F^{\mu\nu} v_\nu = 0. \tag{12}$$

Of course, nontrivial solutions to eq. (4) with $j^\mu = 0$ are still available. Therefore, applying the principle of least action to (1) and using the retarded boundary condition gives the picture involving only free fields. This completes the proof of the theorem which reads: The retarded interaction of the classical Maxwell field with point charges shows up as vanishing in odd-dimensional spacetimes.

Note that this result is insensitive to the form of the regularization prescription in the sense that all reasonable extensions of the definition of the ill-defined expression (8) as a distribution lead inevitably to $A^\mu = \partial^\mu \chi$. By contrast, in the even-dimensional case, no regularization will render the corresponding integral uniformly convergent; in fact, any regularization is superfluous here, which is exemplified most clearly by (11).

Discussion and outlook. – An apparent objection against the above result may sound as follows.

A straight uniformly charged infinite string generates a static two-dimensional field. This setting provides a simple counterexample of the statement that a nontrivial electromagnetic field generated by a point source cannot exist in $(1+2)$-dimensional worlds. However, this objection does not discriminate between a system living in a genuine $(2n+1)$-dimensional realm and that constrained effectively in a $2n$-dimensional spacelike manifold (as with the static field due to the rectilinear charged string, which, owing to its cylindrical symmetry, is generally regarded as a "two-dimensional" field), but, in fact, immersed in a higher-dimensional spacetime.

To appreciate the distinction between "genuine" and "effective", let us compare the behavior of a charged particle in a genuine $(1+1)$-dimensional realm and that in the case that the particle is moving along a straight line in ambient space. In the former case, the field strength contains only the electric component F_{01} but the magnetic field is absent which implies that the Poynting vector is identically zero, so that the accelerated charged particle does not emit radiation [1,2]. On the other hand, a charged particle moving along a straight line in ordinary three-dimensional space emits radiation with the emission rate proportional to the square of its acceleration.

This consideration suggests that the physics of a genuine $(2n+1)$-dimensional realm may differ drastically from that of systems which are constrained in a $2n$-dimensional manifold. Although such systems living in higher-dimensional ambient spaces can imitate some $(2n+1)$-dimensional pictures, their behavior need not be governed by the laws of genuine $(2n+1)$-dimensional realms.

It is interesting that the discussed no-go theorem is inherently classical. Indeed, if we substitute the retardation condition for the Stückelberg-Feynman boundary condition and carry out the Wick rotation in the complex energy plane, then the propagator of a free massless field in quantum theory, defined in a flat D-dimensional spacetime, becomes

$$D_E(x) \propto \frac{1}{(x_E^2)^{\frac{D}{2}-1}}, \tag{13}$$

where x_E^2 is the negatively defined Euclidean length squared of the radius vector x^μ,

$$x_E^2 = -(x_4^2 + \mathbf{x}^2), \quad x_4 = -ix^0. \tag{14}$$

Equation (13) shows that the analytical form of the photon propagator, appearing in perturbative calculations, is common to both $D=2n+1$ and $D=2n$. The basic features of quantum physics in the former case can be obtained from those in the latter case by the analytic continuation in (13).

It may be worth pointing out that Gauss' law holds true in any dimension because it results from the action principle applied to the action (1) with arbitrary D, while the actual coupling of the electromagnetic field and charged matter in the case $D=2n+1$ may occur both trivial and nontrivial according to which boundary condition is additionally imposed.

We finally consider the effect of incorporation of the Chern-Simons term into the $D=3$ electrodynamic action. Let us proceed from the action [7]

$$S = S_{\rm p} - \int {\rm d}^3x \left[j^\mu A_\mu + \frac{1}{8\pi}\left(F_{\mu\nu}F^{\mu\nu} - \mu\,\epsilon^{\alpha\beta\gamma}A_\alpha F_{\beta\gamma}\right)\right]. \tag{15}$$

The field equation which follows from (15) is

$$\partial_\alpha F^{\alpha\beta} + \mu\,{}^*F^\beta = 2\pi j^\alpha, \tag{16}$$

where ${}^*F^\alpha = \frac{1}{2}\,\epsilon^{\alpha\beta\gamma}F_{\beta\gamma}$. We rewrite this equation as

$$\Lambda^{\alpha\beta}(\partial)\,{}^*F_\beta = (\mu\,\eta^{\alpha\beta} + \epsilon^{\alpha\beta\gamma}\partial_\gamma)\,{}^*F_\beta = 2\pi j^\alpha, \tag{17}$$

iterate it with $\Lambda(\partial)$, and use the Bianchi identity $\partial^*_\beta F^\beta = 0$, to obtain

$$(\Box + \mu)\,{}^*F_\alpha = 2\pi\Lambda_{\alpha\beta}(\partial) j^\beta. \tag{18}$$

It is clear from (18), even without explicitly writing its solution, that the field strength generated by a point charged particle is nonzero because eq. (18) allows expressing the field $F_{\alpha\beta}$ directly in terms of the source j^α. We thus see that the classical dynamics governed by the action (15) is nontrivial.

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