

A higher-order tangent map and a conjecture on the higher Nash blowup of curves

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Abstract

We introduce a higher-order version of the tangent map of a morphism and find a matrix representation. We then apply this matrix to solve a conjecture by T. Yasuda regarding the semigroup of the higher Nash blowup of formal curves. We first show that the conjecture is true for toric curves. We conclude by exhibiting a family of non-monomial curves where the conjecture fails.

Keywords: Higher-order tangent map, Nash blowups, Jacobian matrix, curves.

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Introduction

In recent years several authors have introduced higher-order versions of the Jacobian matrix. In [7], a higher-order Jacobian matrix is studied in relation with the higher Nash blowup of a hypersurface. More recently, in [2, 3], a similar matrix is introduced for any finitely generated algebra. In these articles the matrices are used to study singularities in arbitrary characteristic or to study algebraic properties of the module of Kähler differentials of high order. In another but related direction, in [5] it is described a matrix associated to a relative compactification of the induced map on the main components of jet schemes of a projective birational morphism.

In this paper we introduce a matrix that represents a higher-order tangent map of a morphism. This matrix involves higher-order derivatives, making it more suitable for some computations related to jet-spaces. Our main application of this matrix is the solution of a conjecture by T. Yasuda related to the higher Nash blowup of formal curves.

The Nash blowup and the higher Nash blowup of an algebraic variety are modifications that replace singular points by limits of certain vector spaces associated to the variety at non-singular points. There are several questions relating these modifications to resolution of singularities ([15, 18, 22]). Those questions have been treated in [15, 17, 9, 10, 13, 19, 22, 11, 12, 1, 6, 8, 21]. In [22], T. Yasuda proved that the higher Nash blowup solves singularities of curves. In a subsequent paper, Yasuda gave a conjectural explicit description of the semigroup of the higher Nash blowup of formal curves. In this paper we show that the conjecture is true for toric curves but false in general.

To study Yasuda's conjecture in the case of toric curves we first develop a combinatorial description for the higher Nash blowup of toric varieties. This result is based and inspired in the analogous description of the usual

Nash blowup of toric varieties given in [11, 12]. Our results depend strongly on the general framework developed in [11] for not necessarily normal toric varieties. With this general combinatorial description at hand, we are able to prove Yasuda's conjecture for toric curves.

Finally, we present a family of non-monomial curves showing that Yasuda's conjecture fails in general. By combining the results we obtained for monomial morphisms and the general construction of the matrix representing the higher-order tangent map, we are able to describe a particular element of the semigroup of the higher Nash blowup of this family of curves which does not belong to the semigroup suggested by Yasuda.

The present paper is organized as follows. In section 1 we introduce a higher-order tangent map, find a matrix representation and study its basic properties. In section 2 we study in detail the higher-order Jacobian matrix of monomial morphisms. In section 3 we construct a special cover for the higher Nash blowup of a toric variety. Section 4 is devoted to prove Yasuda's conjecture concerning the explicit description of the semigroup of the higher Nash blowup of a toric curve. Finally, in section 5 we exhibit a family of non-monomial curves for which Yasuda's conjecture fails.

Convention: Throughout this paper, \mathbb{K} denotes a field of characteristic zero. In addition, starting at Section 3, we also assume that \mathbb{K} is algebraically closed.

1 A higher-order Jacobian matrix of morphisms

1.1 A higher-order Jacobian matrix of a morphism of affine spaces

In this section we study a higher-order derivative of a morphism between affine varieties and find a matrix representation of this linear map.

Notation 1.1. The following notation will be constantly used in this paper.

- The entries of vectors $\alpha \in \mathbb{N}^t$ are denoted as $\alpha = (\alpha(1), \dots, \alpha(t))$.
- $\alpha \leq \beta \Leftrightarrow \alpha(i) \leq \beta(i) \forall i \in \{1, \dots, t\}$. In particular, $\alpha < \beta$ if and only if $\alpha(i) \leq \beta(i) \forall i \in \{1, \dots, t\}$ and $\alpha(i) < \beta(i)$ for some $i \in \{1, \dots, t\}$.
- $|\alpha| = \alpha(1) + \dots + \alpha(t)$.
- $\alpha! = \alpha(1)! \alpha(2)! \dots \alpha(t)!$

- $\partial^\alpha = \partial^{\alpha(1)}\partial^{\alpha(2)}\dots\partial^{\alpha(t)}$.
- For $t, n \in \mathbb{N}$, $\Lambda_{t,n} := \{\gamma \in \mathbb{N}^t \mid 1 \leq |\gamma| \leq n\}$. In addition, we denote $\lambda_{t,n} := |\Lambda_{t,n}| = \binom{n+t}{t} - 1$.

Consider a morphism

$$\begin{aligned} \varphi : \mathbb{K}^d &\rightarrow \mathbb{K}^s, \\ x = (x_1, \dots, x_d) &\mapsto (g_1(x), \dots, g_s(x)). \end{aligned}$$

Assume that φ is regular at some $x \in \mathbb{K}^d$ and let $y = \varphi(x) \in \mathbb{K}^s$. Let $\mathfrak{m} \subset \mathbb{K}[X_1, \dots, X_d]$ and $\mathfrak{n} \subset \mathbb{K}[Y_1, \dots, Y_s]$ be the maximal ideals corresponding to x and y , and $\mathfrak{m}_x, \mathfrak{n}_y$ the maximal ideals in $(\mathbb{K}[X_1, \dots, X_d])_{\mathfrak{m}}$ and $(\mathbb{K}[Y_1, \dots, Y_s])_{\mathfrak{n}}$, respectively.

Let $\varphi^* : (\mathbb{K}[Y_1, \dots, Y_s])_{\mathfrak{n}} \rightarrow (\mathbb{K}[X_1, \dots, X_d])_{\mathfrak{m}}$ be the induced homomorphism on local rings, where $\varphi^*(\mathfrak{n}_y) \subset \mathfrak{m}_x$. In particular, there is a homomorphism of \mathbb{K} -vector spaces for each $n \in \mathbb{N}$:

$$(\bar{\varphi}^*)_n : \mathfrak{n}_y/\mathfrak{n}_y^{n+1} \rightarrow \mathfrak{m}_x/\mathfrak{m}_x^{n+1}.$$

Let $A_x = \{(X - x)^\alpha := (X_1 - x_1)^{\alpha(1)} \dots (X_d - x_d)^{\alpha(d)} \mid \alpha \in \Lambda_{d,n}\}$ be a basis of $\mathfrak{m}_x/\mathfrak{m}_x^{n+1}$ as \mathbb{K} -vector space. Similarly, let $B_y = \{(Y - y)^\beta \mid \beta \in \Lambda_{s,n}\}$ be a basis of $\mathfrak{n}_y/\mathfrak{n}_y^{n+1}$. The dual bases of A_x and B_y are, respectively,

$$\begin{aligned} A_x^\vee &= \left\{ \frac{1}{\alpha!} \frac{\partial^\alpha}{\partial X^\alpha} \Big|_x \mid \alpha \in \Lambda_{d,n} \right\}, \\ B_y^\vee &= \left\{ \frac{1}{\beta!} \frac{\partial^\beta}{\partial Y^\beta} \Big|_y \mid \beta \in \Lambda_{s,n} \right\}. \end{aligned}$$

Since $(\bar{\varphi}^*)_n((Y - y)^\beta) = (g_1 - g_1(x))^{\beta(1)} \dots (g_s - g_s(x))^{\beta(s)} =: (\varphi - \varphi(x))^\beta$, it follows that the dual morphism $(\bar{\varphi}^*)_n^\vee : (\mathfrak{m}_x/\mathfrak{m}_x^{n+1})^\vee \rightarrow (\mathfrak{n}_y/\mathfrak{n}_y^{n+1})^\vee$ satisfies

$$\begin{aligned} (\bar{\varphi}^*)_n^\vee \left(\frac{1}{\alpha!} \frac{\partial^\alpha}{\partial X^\alpha} \Big|_x \right) &= \frac{1}{\alpha!} \frac{\partial^\alpha}{\partial X^\alpha} \Big|_x \circ (\bar{\varphi}^*)_n : \mathfrak{n}_y/\mathfrak{n}_y^{n+1} \rightarrow \mathbb{K}, \\ (Y - y)^\beta &\mapsto \frac{1}{\alpha!} \frac{\partial^\alpha (\varphi - \varphi(x))^\beta}{\partial X^\alpha} \Big|_x. \end{aligned} \tag{1}$$

It follows that the matrix representation of $(\bar{\varphi}^*)_n^\vee$ in these bases is:

$$[(\bar{\varphi}^*)_n^\vee]_{A_x^\vee}^{B_y^\vee} = \left(\frac{1}{\alpha!} \frac{\partial^\alpha (\varphi - \varphi(x))^\beta}{\partial X^\alpha} \Big|_x \right)_{\beta \in \Lambda_{s,n}, \alpha \in \Lambda_{d,n}}. \tag{2}$$

Definition 1.2. Let $\varphi : \mathbb{K}^d \rightarrow \mathbb{K}^s$ be as before, where $\varphi(x) = y$. We call the linear map $(\bar{\varphi}^*)_n^\vee$ the *derivative of order n of φ at x* . In addition, let

$$D_x^n(\varphi) := \left(\frac{1}{\alpha!} \frac{\partial^\alpha (\varphi - \varphi(x))^\beta}{\partial X^\alpha} \Big|_x \right)_{\beta \in \Lambda_{s,n}, \alpha \in \Lambda_{d,n}}.$$

We call $D_x^n(\varphi)$ the *Jacobian matrix of order n of φ at x* or the *higher-order Jacobian matrix of φ at x* . Notice that $D_x^n(\varphi)$ is a $(\lambda_{s,n} \times \lambda_{d,n})$ -matrix. We order the rows and columns of this matrix increasingly using graded lexicographical order on $\Lambda_{s,n}$ and $\Lambda_{d,n}$. This order is denoted \preceq .

Remark 1.3. Notice that, for each $\beta \in \Lambda_{s,n}$, the β row of $D_x^n(\varphi)$ corresponds precisely to the coefficients of the truncated Taylor expansion of order n of $(\varphi - \varphi(x))^\beta$ centered at x .

Remark 1.4. A similar higher-order Jacobian matrix of a single polynomial F was defined in [7] and is denoted $Jac_n(F)$. See also [2, 3] for a further development of this matrix.

Example 1.5. Let $\varphi : \mathbb{K} \rightarrow \mathbb{K}^2, t \mapsto (t, t^2)$. The usual matrix representation of the derivative of φ at $0 \in \mathbb{K}$ is given by the Jacobian matrix:

$$D_0(\varphi) = \left(\begin{array}{c} \frac{dt}{dt} \Big|_0 \\ \frac{dt^2}{dt} \Big|_0 \end{array} \right) = \left(\begin{array}{c} 1 \\ 2t \end{array} \right) \Big|_0.$$

Following the construction of the higher-order Jacobian matrix given previously, in the case $n = 2$, we obtain:

$$D_0^2(\varphi) = \left(\begin{array}{cc} \frac{dt}{dt} \Big|_0 & \frac{1}{2!} \frac{d^2 t}{dt^2} \Big|_0 \\ \frac{dt^2}{dt} \Big|_0 & \frac{1}{2!} \frac{d^2 t^2}{dt^2} \Big|_0 \\ \frac{d(t^2)}{dt} \Big|_0 & \frac{1}{2!} \frac{d^2 (t^2)^2}{dt^2} \Big|_0 \\ \frac{d(t \cdot t^2)}{dt} \Big|_0 & \frac{1}{2!} \frac{d^2 (t \cdot t^3)}{dt^2} \Big|_0 \\ \frac{d(t^2)^2}{dt} \Big|_0 & \frac{1}{2!} \frac{d^2 (t^2)^2}{dt^2} \Big|_0 \end{array} \right) = \left(\begin{array}{cc} 1 & 0 \\ 2t & 1 \\ 0 & 1 \\ 0 & 2t \\ 0 & 4t^2 \end{array} \right) \Big|_0.$$

The higher-order Jacobian matrix satisfies the following basic properties.

Lemma 1.6. Let $\varphi : \mathbb{K}^d \rightarrow \mathbb{K}^s$ be as before.

(i) If $\beta \in \Lambda_{s,n}$ is such that $|\beta| = 1$ then $\frac{\partial^\alpha (\varphi - \varphi(x))^\beta}{\partial X^\alpha} = \frac{\partial^\alpha \varphi^\beta}{\partial X^\alpha}$ for every $\alpha \in \Lambda_{d,n}$.

(ii) Let $\alpha \in \Lambda_{d,n}, \beta \in \Lambda_{s,n}$ be such that $|\alpha| < |\beta|$. Then $\frac{\partial^\alpha (\varphi - \varphi(x))^\beta}{\partial X^\alpha} \Big|_x = 0$.

(iii) $D_x^1(\varphi)$ is the usual Jacobian of φ evaluated at x .

(iv) If $\varphi : \mathbb{K}^d \rightarrow \mathbb{K}^d$ is the identity then $D_x^n(\varphi)$ is the identity matrix.

(v) Let $\psi : \mathbb{K}^s \rightarrow \mathbb{K}^r$ be another morphism. Then $D_x^n(\psi \circ \varphi) = D_y^n(\psi)D_x^n(\varphi)$.

Proof. (i) If $\beta \in \Lambda_{s,n}$ is such that $|\beta| = 1$ then $(\varphi - \varphi(x))^\beta = g_i - g_i(x)$ for some $i \in \{1, \dots, s\}$. Since $g_i(x)$ is a constant, the result follows.

(ii) The hypothesis on α and β means that in $\frac{\partial^\alpha(\varphi - \varphi(x))^\beta}{\partial X^\alpha}$ the order of the derivative is less than the number of factors in $(\varphi - \varphi(x))^\beta$. This implies that in every summand of $\frac{\partial^\alpha(\varphi - \varphi(x))^\beta}{\partial X^\alpha}$ there is a factor $g_i - g_i(x)$. Thus $\frac{\partial^\alpha(\varphi - \varphi(x))^\beta}{\partial X^\alpha}|_x = 0$.

(iii) This follows from the definition of $D_x^n(\varphi)$ and (i).

(iv) If φ is the identity then $(\bar{\varphi}^*)_n : \mathfrak{m}_x/\mathfrak{m}_x^{n+1} \rightarrow \mathfrak{m}_x/\mathfrak{m}_x^{n+1}$ is also the identity. By choosing the same basis for both vector spaces we conclude that $D_x^n(\varphi)$ is the identity matrix.

(v) We know that $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$. Thus, $((\psi \circ \varphi)^*)_n = (\bar{\varphi}^*)_n \circ (\bar{\psi}^*)_n$. Taking duals $((\psi \circ \varphi)^*)_n^\vee = ((\bar{\varphi}^*)_n \circ (\bar{\psi}^*)_n)^\vee = (\bar{\psi}^*)_n^\vee \circ (\bar{\varphi}^*)_n^\vee$. The result follows. \square

Now suppose that $X \subset \mathbb{K}^d$ and $Y \subset \mathbb{K}^s$ are affine varieties and let $\varphi : X \rightarrow Y$ be a morphism which is regular at $x \in X$ and let $y = \varphi(x)$. Denote by $\bar{\mathfrak{m}}_x$ and $\bar{\mathfrak{n}}_y$ the maximal ideals of the corresponding local rings. Since φ is the restriction of a morphism $\varphi : \mathbb{K}^d \rightarrow \mathbb{K}^s$, the diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \downarrow i & & \downarrow i \\ \mathbb{K}^d & \xrightarrow{\varphi} & \mathbb{K}^s \end{array}$$

induces the diagram

$$\begin{array}{ccc} (\bar{\mathfrak{m}}_x/\bar{\mathfrak{m}}_x^{n+1})^\vee & \longrightarrow & (\bar{\mathfrak{n}}_y/\bar{\mathfrak{n}}_y^{n+1})^\vee \\ \downarrow & & \downarrow \\ (\mathfrak{m}_x/\mathfrak{m}_x^{n+1})^\vee & \longrightarrow & (\mathfrak{n}_y/\mathfrak{n}_y^{n+1})^\vee \end{array}$$

Taking bases as before we identify $(\bar{\mathfrak{m}}_x/\bar{\mathfrak{m}}_x^{n+1})^\vee \cong \mathbb{K}^{\lambda_{d,n}}$ and $(\bar{\mathfrak{n}}_y/\bar{\mathfrak{n}}_y^{n+1})^\vee \cong \mathbb{K}^{\lambda_{s,n}}$. The commutativity of the diagram

$$\begin{array}{ccc} (\bar{\mathfrak{m}}_x/\bar{\mathfrak{m}}_x^{n+1})^\vee & \longrightarrow & (\bar{\mathfrak{n}}_y/\bar{\mathfrak{n}}_y^{n+1})^\vee \\ \downarrow & & \downarrow \\ \mathbb{K}^{\lambda_{d,n}} & \xrightarrow{D_x^n(\varphi)} & \mathbb{K}^{\lambda_{s,n}} \end{array}$$

allows us to define a higher-order tangent map of $\varphi : X \rightarrow Y$ at $x \in X$ as the restriction

$$D_x^n(\varphi) : (\bar{\mathfrak{m}}_x/\bar{\mathfrak{m}}_x^{n+1})^\vee \rightarrow (\bar{\mathfrak{n}}_y/\bar{\mathfrak{n}}_y^{n+1})^\vee.$$

1.2 Higher-order Jacobian matrix and birational morphisms

Let $Y \subset \mathbb{K}^s$ be an irreducible algebraic variety and $y \in Y$. In this subsection, we use the higher-order Jacobian matrix to explicitly compute the space $(\bar{\mathfrak{n}}_y/\bar{\mathfrak{n}}_y^{n+1})^\vee$ in some cases.

Lemma 1.7. *Let X and Y be irreducible varieties and let $\varphi : X \dashrightarrow Y$ be a birational morphism. Let $U \subset X$ and $V \subset Y$ be isomorphic open subsets. Let $x \in U$ and $y = \varphi(x) \in V$. Then φ induces an isomorphism $\bar{\mathfrak{n}}_y/\bar{\mathfrak{n}}_y^{n+1} \cong \bar{\mathfrak{m}}_x/\bar{\mathfrak{m}}_x^{n+1}$.*

Proof. Since $\varphi|_U : U \rightarrow V$ is an isomorphism, there is an induced isomorphism on local rings $\mathcal{O}_{Y,y} \cong \mathcal{O}_{X,x}$. In particular, $\varphi^*(\bar{\mathfrak{n}}_y) = \bar{\mathfrak{m}}_x$. The result follows. \square

Proposition 1.8. *Let $\varphi : \mathbb{K}^d \dashrightarrow Y \subset \mathbb{K}^s$ be a birational morphism, $U \subset \mathbb{K}^d$ and $V \subset Y$ isomorphic open subsets, and $y = \varphi(x)$ for some $x \in U$. Then the vector space $(\bar{\mathfrak{n}}_y/\bar{\mathfrak{n}}_y^{n+1})^\vee$ is isomorphic to the image of the linear map defined by $D_x^n(\varphi)$. In particular, $\text{rank}(D_x^n(\varphi)) = \lambda_{d,n}$.*

Proof. We have the following commutative diagram

$$\begin{array}{ccc} \mathbb{K}^d & \xrightarrow{i \circ \varphi} & \mathbb{K}^s \\ & \searrow \varphi & \uparrow i \\ & & Y \end{array}$$

This diagram induces in turn the following commutative diagram

$$\begin{array}{ccc}
(\mathfrak{m}_x/\mathfrak{m}_x^{n+1})^\vee & \xrightarrow{\overline{(i \circ \varphi)}^*} & (\mathfrak{n}_y/\mathfrak{n}_y^{n+1})^\vee \\
\searrow \cong & & \uparrow (i^*)^\vee \\
& & (\bar{\mathfrak{n}}_y/\bar{\mathfrak{n}}_y^{n+1})^\vee \\
& \swarrow (\bar{\varphi}^*)^\vee &
\end{array}$$

where the isomorphism in the diagonal arrow comes from lemma 1.7. Fixing bases for $\mathfrak{m}_x/\mathfrak{m}_x^{n+1}$ and $\mathfrak{n}_y/\mathfrak{n}_y^{n+1}$ as in the previous section, we identify $(\mathfrak{m}_x/\mathfrak{m}_x^{n+1})^\vee \cong \mathbb{K}^{\lambda_{d,n}}$ and $(\mathfrak{n}_y/\mathfrak{n}_y^{n+1})^\vee \cong \mathbb{K}^{\lambda_{s,n}}$. In addition, from (2), it follows that $\overline{(i \circ \varphi)}^*$ is the linear map defined by the matrix $D_x^n(i \circ \varphi) = D_x^n(\varphi)$. We thus obtain the diagram

$$\begin{array}{ccc}
\mathbb{K}^{\lambda_{d,n}} & \xrightarrow{D_x^n(\varphi)} & \mathbb{K}^{\lambda_{s,n}} \\
\searrow & & \uparrow \\
& & (\bar{\mathfrak{n}}_y/\bar{\mathfrak{n}}_y^{n+1})^\vee.
\end{array}$$

The commutativity of this diagram proves the proposition. \square

Remark 1.9. Notice that the proofs in this section considers local rings of points of a variety. Therefore, these results are also valid in the analytic case. In particular, we can define a higher-order Jacobian matrix for germs of analytic maps $\varphi : (X, x) \rightarrow (Y, y)$.

Example 1.10. Let $\varphi : \mathbb{K} \rightarrow C = \mathbf{V}(y - x^2) \subset \mathbb{K}^2$, $t \mapsto (t, t^2)$. We computed $D_0^2(\varphi) : \mathbb{K}^2 \rightarrow \mathbb{K}^5$ in the previous section. Let $\bar{\mathfrak{n}}_0$ be the maximal ideal of $(0, 0) \in C$. Using proposition 1.8 we obtain

$$(\bar{\mathfrak{n}}_0/\bar{\mathfrak{n}}_0^3)^\vee = \text{Im}(D_0^2(\varphi)) = \text{Im} \left(\begin{array}{cc} 1 & 0 \\ 2t & 1 \\ 0 & 1 \\ 0 & 2t \\ 0 & 4t^2 \end{array} \right) \Big|_0 \subset \mathbb{K}^5.$$

2 Higher order Jacobian matrix for monomial morphisms

Let $a_1, \dots, a_s \in \mathbb{Z}^d$. We assume that $d \leq s$. In this section we study the higher-order Jacobian matrix of the monomial morphism

$$\begin{aligned} \varphi : (\mathbb{K} \setminus \{0\})^d &\rightarrow \mathbb{K}^s \\ x = (x_1, \dots, x_d) &\mapsto (x^{a_1}, \dots, x^{a_s}), \end{aligned} \quad (3)$$

where $x^{a_i} := x_1^{a_i(1)} \cdots x_d^{a_i(d)}$.

Notation 2.1. The following notation will be also used constantly.

- A denotes the $(d \times s)$ -matrix whose columns are the vectors a_1, \dots, a_s . By abuse of notation, the set $\{a_1, \dots, a_s\}$ is also denoted as A .
- $A_i := (a_1(i), \dots, a_s(i))$, $i = 1, \dots, d$, denote the rows of A . In particular, for $\gamma \in \mathbb{N}^s$,

$$X^{A\gamma} = X_1^{A_1 \cdot \gamma} \cdots X_d^{A_d \cdot \gamma},$$

where $A\gamma$ is a product of matrices and $A_i \cdot \gamma$ is the usual inner product in \mathbb{R}^s .

- For $\beta \in \mathbb{N}^s$, we denote

$$(X^A - x^A)^\beta := (X^{a_1} - x^{a_1})^{\beta(1)} \cdots (X^{a_s} - x^{a_s})^{\beta(s)}.$$

- For $\lambda, \tau \in \mathbb{N}^t$, denote $\binom{\lambda}{\tau} := \binom{\lambda(1)}{\tau(1)} \cdots \binom{\lambda(t)}{\tau(t)}$.

With this notation, the higher-order Jacobian of φ at a point $x \in (\mathbb{K} \setminus \{0\})^d$ is given by:

$$D_x^n(\varphi) = \left(\frac{1}{\alpha!} \frac{\partial^\alpha (X^A - x^A)^\beta}{\partial X^\alpha} \Big|_x \right)_{\beta \in \Lambda_{s,n}, \alpha \in \Lambda_{d,n}}.$$

We are interested in computing the maximal minors of this matrix. This will be done in several steps.

Lemma 2.2. *Let $\gamma \in \mathbb{N}^s$ and $\alpha \in \mathbb{N}^d$. Then*

$$\frac{1}{\alpha!} \frac{\partial^\alpha (X^{A\gamma})}{\partial X^\alpha} = \binom{A\gamma}{\alpha} X^{A\gamma - \alpha}.$$

Proof. This is a direct computation. □

Lemma 2.3. Let $\beta \in \Lambda_{s,n}$, $\alpha \in \Lambda_{d,n}$ and $x \in (\mathbb{K} \setminus \{0\})^d$. Then

$$\frac{1}{\alpha!} \partial^\alpha (X^A - x^A)^\beta|_x = c_{\beta,\alpha} x^{A\beta-\alpha},$$

where $c_{\beta,\alpha} := \sum_{\gamma \leq \beta, \gamma \neq 0} (-1)^{|\beta-\gamma|} \binom{\beta}{\gamma} \binom{A\gamma}{\alpha}$.

Proof. From the binomial theorem we obtain, for each $i \in \{1, \dots, s\}$:

$$(X^{a_i} - x^{a_i})^{\beta(i)} = \sum_{\gamma(i)=0}^{\beta(i)} (-1)^{\beta(i)-\gamma(i)} \binom{\beta(i)}{\gamma(i)} (X^{a_i})^{\gamma(i)} (x^{a_i})^{\beta(i)-\gamma(i)}.$$

Thus, letting $\gamma := (\gamma(1), \dots, \gamma(s))$,

$$\begin{aligned} (X^A - x^A)^\beta &= \sum_{\gamma(1)=0}^{\beta(1)} \dots \sum_{\gamma(s)=0}^{\beta(s)} (-1)^{|\beta-\gamma|} \binom{\beta}{\gamma} \prod_{i=1}^s (X^{a_i})^{\gamma(i)} (x^{a_i})^{\beta(i)-\gamma(i)} \\ &= \sum_{\gamma \leq \beta} (-1)^{|\beta-\gamma|} \binom{\beta}{\gamma} (X^{\sum \gamma(i)a_i}) (x^{\sum \beta(i)a_i - \sum \gamma(i)a_i}) \\ &= \sum_{\gamma \leq \beta} (-1)^{|\beta-\gamma|} \binom{\beta}{\gamma} (X^{A\gamma}) (x^{A\beta-A\gamma}). \end{aligned}$$

With this formula and the previous lemma now it is easy to compute the derivative evaluated at x :

$$\begin{aligned} \frac{1}{\alpha!} \partial^\alpha (X^A - x^A)^\beta|_x &= \sum_{\gamma \leq \beta, \gamma \neq 0} (-1)^{|\beta-\gamma|} \binom{\beta}{\gamma} (x^{A\beta-A\gamma}) \frac{1}{\alpha!} \partial^\alpha (X^{A\gamma})|_x \\ &= \sum_{\gamma \leq \beta, \gamma \neq 0} (-1)^{|\beta-\gamma|} \binom{\beta}{\gamma} (x^{A\beta-A\gamma}) \binom{A\gamma}{\alpha} X^{A\gamma-\alpha}|_x \\ &= \sum_{\gamma \leq \beta, \gamma \neq 0} (-1)^{|\beta-\gamma|} \binom{\beta}{\gamma} \binom{A\gamma}{\alpha} x^{A\beta-\alpha} \\ &= \left[\sum_{\gamma \leq \beta, \gamma \neq 0} (-1)^{|\beta-\gamma|} \binom{\beta}{\gamma} \binom{A\gamma}{\alpha} \right] x^{A\beta-\alpha}. \end{aligned}$$

□

Using this lemma it follows that the higher-order Jacobian of φ at each $x \in (\mathbb{K} \setminus \{0\})^d$ has the following shape:

$$D_x^n(\varphi) = \left(c_{\beta,\alpha} x^{A\beta-\alpha} \right)_{\beta \in \Lambda_{s,n}, \alpha \in \Lambda_{d,n}}. \quad (4)$$

Proposition 2.4. Let $J = \{\beta_1, \dots, \beta_{\lambda_{d,n}}\} \subset \Lambda_{s,n}$, where $\beta_1 \prec \dots \prec \beta_{\lambda_{d,n}}$ (see definition 1.2 for the notation \prec). Let L_J denote the submatrix of $D_x^n(\varphi)$ formed by the rows $\beta_1, \dots, \beta_{\lambda_{d,n}}$ and all of its columns $\alpha_1, \dots, \alpha_{\lambda_{d,n}}$. Then, if $x \in (\mathbb{K} \setminus \{0\})^d$,

$$\det(L_J) = \frac{x^{A\beta_1 + \dots + A\beta_{\lambda_{d,n}}}}{x^{\alpha_1 + \dots + \alpha_{\lambda_{d,n}}}} \det(L_J^c),$$

where $L_J^c := (c_{\beta_i, \alpha_j})_{i,j}$.

Proof. The matrix whose determinant we want to compute is the following:

$$L_J = \begin{pmatrix} c_{\beta_1, \alpha_1} x^{A\beta_1 - \alpha_1} & \dots & c_{\beta_1, \alpha_{\lambda_{d,n}}} x^{A\beta_1 - \alpha_{\lambda_{d,n}}} \\ c_{\beta_2, \alpha_1} x^{A\beta_2 - \alpha_1} & \dots & c_{\beta_2, \alpha_{\lambda_{d,n}}} x^{A\beta_2 - \alpha_{\lambda_{d,n}}} \\ \vdots & \dots & \vdots \\ c_{\beta_{\lambda_{d,n}}, \alpha_1} x^{A\beta_{\lambda_{d,n}} - \alpha_1} & \dots & c_{\beta_{\lambda_{d,n}}, \alpha_{\lambda_{d,n}}} x^{A\beta_{\lambda_{d,n}} - \alpha_{\lambda_{d,n}}} \end{pmatrix}.$$

Multiply the α_j th column by x^{α_j} . Then multiply the β_i th row by $x^{-A\beta_i}$. Let $L_J^c = (c_{\beta_i, \alpha_j})_{i,j}$. Then

$$\det(L_J) = \frac{x^{A\beta_1 + \dots + A\beta_{\lambda_{d,n}}}}{x^{\alpha_1 + \dots + \alpha_{\lambda_{d,n}}}} \det(L_J^c). \quad (5)$$

□

Remark 2.5. If $n = 1$ then $\lambda_{d,n} = d$. Letting $\beta_{i_k} = e_{i_k} \in \mathbb{N}^s$ for $k = 1, \dots, d$ and $J = \{\beta_{i_1}, \dots, \beta_{i_d}\} \subset \Lambda_{s,1}$, it follows that L_J^c is the $(d \times d)$ -matrix whose rows are a_{i_1}, \dots, a_{i_d} . In particular, in view of (5), $\det(L_J) \neq 0$ if and only if a_{i_1}, \dots, a_{i_d} are linearly independent. This remark allows a comparison between the so-called logarithmic Jacobian ideal of a toric variety and an ideal whose blowup defines the Nash blowup of the variety [9, 15]. This, in turn, gives place to the fact that the Nash blowup of a toric variety can be obtained as the blowup of its logarithmic Jacobian ideal (see [9, 14, 11]). As a result, there is an explicit combinatorial description of the Nash blowup in this context [9, 11, 12].

3 Higher Nash blowup of toric varieties

In this section we exhibit an open cover for the higher Nash blowup of a toric variety. The main result is the first step toward our study of a conjecture proposed by T. Yasuda regarding the higher Nash blowup of formal curves.

We start by recalling the definition of the higher Nash blowup of an algebraic variety. Subsection 3.2 is based on the general theory of (not necessarily normal) toric varieties developed in [11] and also uses some ideas appearing in [12].

3.1 Higher Nash blowup

Notation 3.1. Given an irreducible algebraic variety $X \subset \mathbb{K}^s$ of dimension d and a point $x \in X$, we denote $T_x^n X := (\bar{\mathfrak{m}}_x / \bar{\mathfrak{m}}_x^{n+1})^\vee$. This is a vector space of dimension $\lambda_{d,n}$, whenever x is a non-singular point.

Notice that $X \subset \mathbb{K}^s$ implies $T_x^n X \subset T_x^n \mathbb{K}^s \cong \mathbb{K}^{\lambda_{s,n}}$. Thus, if x is a non-singular point, we can see $T_x^n X$ as an element of the Grassmanian $Gr(\lambda_{d,n}, \mathbb{K}^{\lambda_{s,n}})$.

Definition 3.2. [15, 16, 22] Let $X \subset \mathbb{K}^s$ be an irreducible algebraic variety of dimension d . Consider the *Gauss map of order n* :

$$\begin{aligned} G_n : X \setminus \text{Sing}(X) &\rightarrow Gr(\lambda_{d,n}, \mathbb{K}^{\lambda_{s,n}}) \\ x &\mapsto T_x^n X, \end{aligned}$$

where $\text{Sing}(X)$ denotes the set of singular points of X . Denote by $Nash_n(X)$ the Zariski closure of the graph of G_n . Call π_n the restriction to $Nash_n(X)$ of the projection of $X \times Gr(\lambda_{d,n}, \mathbb{K}^{\lambda_{s,n}})$ to X . The pair $(Nash_n(X), \pi_n)$ is called the *higher Nash blowup* of X or the *Nash blowup of X of order n* .

It was proposed by T. Yasuda ([22]) to solve the singularities of X by applying once the higher Nash blowup for n sufficiently large. Yasuda himself proved that his method works for curves ([22, Corollary 3.7]). Moreover, Yasuda suggested in [24, Remark 1.5] that the A_3 -singularity might be a counterexample to his conjecture on the one-step resolution. R. Toh-Yama recently proved in [21] that $Nash_n(A_3)$ is singular for every $n \geq 1$.

In addition to the previous conjecture, Yasuda also proposed another one concerning the numerical semigroup associated to the higher Nash blowup of formal curves (see conjecture 4.2 below). The results we obtain in this section will be used to study this conjecture in the case of toric curves.

3.2 An explicit open cover of the higher Nash blowup of toric varieties by affine toric varieties

Let us recall the definition of an affine toric variety (see, for instance, [4, Section 1.1] or [20, Chapter 4]).

Definition 3.3. Let $A = \{a_1, \dots, a_s\} \subset \mathbb{Z}^d$. Let $\Gamma := \mathbb{N}A$ denote the semi-group generated by A , i.e., $\Gamma = \{\sum_i \lambda_i a_i \mid \lambda_i \in \mathbb{N}\}$. In addition, assume that $\mathbb{Z}A = \{\sum_i \lambda_i a_i \mid \lambda_i \in \mathbb{Z}\} = \mathbb{Z}^d$. Consider the following monomial morphism:

$$\begin{aligned} \varphi_A : (\mathbb{K}^*)^d &\rightarrow \mathbb{K}^s \\ x = (x_1, \dots, x_d) &\mapsto (x^{a_1}, \dots, x^{a_s}), \end{aligned} \tag{6}$$

where $\mathbb{K}^* = \mathbb{K} \setminus \{0\}$. Let X_Γ denote the Zariski closure of the image of φ_A . We call X_Γ the *affine toric variety* defined by Γ .

It is well known that X_Γ is an irreducible variety of dimension d , contains a dense open set isomorphic to $(\mathbb{K}^*)^d$ and such that the natural action of $(\mathbb{K}^*)^d$ on itself extends to an action on the variety. In addition, X_Γ does not depend on the generating set A (see [4, Theorem 1.1.17] for various equivalent characterizations of affine toric varieties).

Proposition 3.4. [4, Prop. 1.2.12], [11, Prop. 15] *Let $X_\Gamma \subset \mathbb{K}^s$ be an affine toric variety, $\sigma^\vee := \mathbb{R}_{\geq 0}\Gamma \subset \mathbb{R}^d$ the cone generated by Γ , and σ its dual cone. The following statements are equivalent:*

- (a) $0 \in X_\Gamma$.
- (b) X_Γ has a 0-dimensional orbit.
- (c) The cone σ is of dimension d .
- (d) The cone σ^\vee is strongly convex.

We want to show that the higher Nash blowup of a toric variety having a 0-dimensional orbit, has a finite open cover given by affine toric varieties with the same property. The proof of this fact is based on the following combinatorial construction of blowing ups of monomials ideals in toric varieties (see [11, Section 2.6]).

Combinatorial description of the blowup of a monomial ideal.

Let $X_\Gamma \subset \mathbb{K}^s$ be an affine toric variety having a 0-dimensional orbit and $\sigma^\vee = \mathbb{R}_{\geq 0}\Gamma \subset \mathbb{R}^d$ (which is strongly convex, by the previous proposition).

- (i) Let $I = \langle x^m \mid m \in B \rangle \subset \mathbb{K}[X_\Gamma]$ be a monomial ideal.
- (ii) Let $\mathcal{N}(I)$ be the Newton polyhedron of I , i.e., the convex hull in \mathbb{R}^d of the set $\{m + \sigma^\vee \mid m \in B\}$.
- (iii) Let $m' \in B$. Denote $\Gamma_{m'} := \Gamma + \mathbb{N}(\{m - m' \mid m \in B\})$.

- (iv) Given $m', m'' \in B$, the affine toric varieties $X_{\Gamma_{m'}}$ and $X_{\Gamma_{m''}}$ can be glued together along the principal open subsets $X_{\Gamma_{m'}} \setminus \mathbf{V}(x^{m''-m'})$ and $X_{\Gamma_{m''}} \setminus \mathbf{V}(x^{m'-m''})$. There is an isomorphism between these open subsets which is induced by localizations of coordinate rings:

$$\mathbb{K}[X_{\Gamma_{m'}}]_{\frac{x^{m''}}{x^{m'}}} \cong \mathbb{K}[X_{\Gamma_{m''}}]_{\frac{x^{m'}}{x^{m''}}}.$$

- (v) The variety resulting from the previous glueing is the blowup of X_Γ along I (see [11, Proposition 32]). We denote it as $Bl_I X_\Gamma$.
- (vi) Finally, let $B' = \{m' \in B \mid m' \text{ is a vertex of } \mathcal{N}(I)\}$. Then

$$Bl_I X_\Gamma = \bigsqcup_{m' \in B'} X_{\Gamma_{m'}} / \sim$$

(see the proof of Proposition 32, [11]). By proposition 3.4, for $m' \in B'$, $X_{\Gamma_{m'}}$ has a 0-dimensional orbit. In particular, $Bl_I X_\Gamma$ has an open cover by affine toric varieties having a 0-dimensional orbit.

Remark 3.5. The variety resulting from the previous construction is an example of an abstract toric variety having a good action (see [11, Section 2.8]). These varieties are characterized by the fact that they can be described in combinatorial terms by families of semigroups labeled by fans (see [11, Theorem 44]).

In order to use the previous construction and compare it to the higher Nash blowup of a toric variety, we need to introduce some monomial ideal. In addition, we use the Plücker embedding of $Gr(\lambda_{d,n}, \mathbb{K}^{\lambda_{s,n}})$ into the projective space $\mathbb{P}^{\binom{\lambda_{s,n}}{\lambda_{d,n}}-1}$. First, some notation.

Notation 3.6. Let $A = \{a_1, \dots, a_s\} \subset \mathbb{Z}^d$ and $\Gamma = \mathbb{N}A$ a semigroup defining a toric variety $X_\Gamma \subset \mathbb{K}^s$.

- Given $J = \{\beta_1, \dots, \beta_{\lambda_{d,n}}\} \subset \Lambda_{s,n}$ such that $\beta_1 \prec \dots \prec \beta_{\lambda_{d,n}}$, we denote as U_J the affine chart of $\mathbb{P}^{\binom{\lambda_{s,n}}{\lambda_{d,n}}-1}$ where the J -coordinate is non-zero (see definition 1.2 for the notation \prec).
- Let $S_A := \{J = \{\beta_1, \dots, \beta_{\lambda_{d,n}}\} \subset \Lambda_{s,n} \mid \beta_1 \prec \dots \prec \beta_{\lambda_{d,n}}, \det(L_J^c) \neq 0\}$. Notice that $S_A \neq \emptyset$ by propositions 1.8 and 2.4.
- For each $J = \{\beta_1, \dots, \beta_{\lambda_{d,n}}\} \subset \Lambda_{s,n}$, denote $m_J := A\beta_1 + \dots + A\beta_{\lambda_{d,n}}$.

Definition 3.7. Let $\mathcal{I}_n := \langle X^{m_J} | J \in S_A \rangle \subset \mathbb{K}[X_\Gamma]$. Following the usual terminology, we call \mathcal{I}_n the *logarithmic Jacobian ideal of order n* of X_Γ .

Remark 3.8. In the following subsection we show that \mathcal{I}_n does not depend on the set of generators of Γ .

We want to apply the combinatorial description of the blowup of a monomial ideal to \mathcal{I}_n . To that end, we simplify a little the notation coming from that description. For $X^{m_J} \in \mathcal{I}_n$, instead of using Γ_{m_J} as in (iii), we simply write Γ_J .

Now we are ready to prove the main theorem of this section.

Theorem 3.9. *Let $X_\Gamma \subset \mathbb{K}^s$ be an affine toric variety having a 0-dimensional orbit. Then $Nash_n(X_\Gamma)$ is isomorphic to the blowup of the logarithmic Jacobian ideal of order n of X_Γ . In particular, $Nash_n(X_\Gamma)$ has a finite open covering given by affine toric varieties having a 0-dimensional orbit.*

Proof. We divide this proof into two steps: the first one describes locally $Nash_n(X_\Gamma)$ and the second one is a glueing argument.

Step I: According to proposition 1.8 and (4), for a point $p := \varphi_A(x) \in X_\Gamma$, for some $x \in (\mathbb{K}^*)^d$, we have

$$T_p^n X_\Gamma = \text{Im}(D_x^n(\varphi_A)) = \text{Im}\left(c_{\beta,\alpha} x^{A\beta-\alpha}\right)_{\beta \in \Lambda_{s,n}, \alpha \in \Lambda_{d,n}}.$$

Thus, the Plücker coordinates of $T_p^n X_\Gamma \in Gr(\lambda_{d,n}, \mathbb{K}^{\lambda_{s,n}}) \hookrightarrow \mathbb{P}^{\binom{\lambda_{s,n}}{\lambda_{d,n}}-1}$ are given by the maximal minors of $(c_{\beta,\alpha} x^{A\beta-\alpha})_{\beta,\alpha}$. According to (5), for a choice $J = \{\beta_1, \dots, \beta_{\lambda_{d,n}}\} \subset \Lambda_{s,n}$, where $\beta_1 \prec \dots \prec \beta_{\lambda_{d,n}}$, the corresponding minor is:

$$\det(L_J^c) \frac{x^{A\beta_1 + \dots + A\beta_{\lambda_{d,n}}}}{x^{\alpha_1 + \dots + \alpha_{\lambda_{d,n}}}}.$$

Fix $J_0 \in S_A$. It follows that:

1. If $J \in S_A$ we can make a change of coordinates in $U_{J_0} \cong \mathbb{K}^{s + \binom{\lambda_{s,n}}{\lambda_{d,n}} - 1}$ to turn the non-zero constant $\frac{\det(L_J^c)}{\det(L_{J_0}^c)}$ into 1. Thus, we can assume that the J -coordinate of $Nash_n(X_\Gamma) \cap U_{J_0}$ comes with the constant 1, for every $J \in S_A$.
2. If $J \notin S_A$ the J -coordinate of $Nash_n(X_\Gamma)$ is zero. This implies that we can embed $Nash_n(X_\Gamma) \cap U_{J_0}$ in $\mathbb{K}^{s + |S_A| - 1}$.

These two remarks imply that

$$\begin{aligned}
Nash_n(X_\Gamma) \cap U_{J_0} &\cong \overline{\left\{ \left(\varphi_A(x), \frac{x^{\sum_{\beta_i \in J} A\beta_i}}{x^{\sum_{\beta_i^0 \in J_0} A\beta_i^0}} \right) \mid J \in S_A \setminus \{J_0\}, x \in (\mathbb{K}^*)^d \right\}} \\
&= \overline{\left\{ (\varphi_A(x), x^{m_J - m_{J_0}}) \mid J \in S_A \setminus \{J_0\}, x \in (\mathbb{K}^*)^d \right\}} \quad (7) \\
&= \overline{Im(\varphi_{\Gamma_{J_0}})} \subset \mathbb{K}^{s+|S_A|-1}.
\end{aligned}$$

In particular, this affine chart of $Nash_n(X_\Gamma)$ is an affine toric variety.

Step II: By Step I, for each $J \in S_A$, $X_{\Gamma_J} \cong Nash_n(X_\Gamma) \cap U_J$. Since both $Bl_{\mathcal{I}_n} X_\Gamma$ and $Nash_n(X_\Gamma)$ are obtained by glueing X_{Γ_J} and $Nash_n(X_\Gamma) \cap U_J$, respectively, we only need to check that the glueing is the same. The glueing in $Nash_n(X_\Gamma) \subset X_\Gamma \times \mathbb{P}^{(\lambda_{d,n}^{s,n})^{-1}}$ is given by the usual glueing in $\mathbb{P}^{(\lambda_{d,n}^{s,n})^{-1}}$, i.e., the one induced by the following isomorphisms of localizations of coordinate rings for each couple $J_1, J_2 \in S_A$:

$$\begin{aligned}
&\mathbb{K}[x^{a_1}, \dots, x^{a_s}, x^{m_J - m_{J_1}} \mid J \in S_A \notin \{J_1\}] \frac{x^{m_{J_2}}}{x^{m_{J_1}}} \\
&\cong \mathbb{K}[x^{a_1}, \dots, x^{a_s}, x^{m_J - m_{J_2}} \mid J \in S_A \notin \{J_2\}] \frac{x^{m_{J_1}}}{x^{m_{J_2}}}.
\end{aligned}$$

This is exactly the glueing described in the combinatorial description of the blowup of a monomial ideal. \square

Remark 3.10. For $n = 1$, the previous theorem was proved in [9, 14, 11].

Remark 3.11. The previous theorem and its proof show that $Nash_n(X_\Gamma)$ can be covered by open affine varieties which are invariant under the action of a torus. This statement could be obtained directly using results of [11, 22]. Indeed, by [22, Section 2.2], the higher Nash blowup of a toric variety is an equivariant morphism; in particular, it is the blowup of some monomial ideal. Then [11, Corollary 34] implies the statement. We want to emphasize that the contribution of this section is that one can take the logarithmic Jacobian ideal of order n as such monomial ideal. In addition, we describe an explicit method to construct this ideal.

3.3 The logarithmic Jacobian ideal of order n is independent of the generators of Γ

In this subsection we show that the ideal \mathcal{I}_n does not depend on the set of generators A of Γ . To that end, we need to modify temporarily the notation

\mathcal{I}_n . We denote as \mathcal{I}_C the logarithmic Jacobian ideal of order n , where C is an arbitrary set of generators of Γ .

Theorem 3.12. *Let $A = \{a_1, \dots, a_s\} \subset \mathbb{Z}^d$ and $B = \{b_1, \dots, b_t\} \subset \mathbb{Z}^d$ be such that $\Gamma = \mathbb{N}A = \mathbb{N}B$. Then $\mathcal{I}_A = \mathcal{I}_{A \cup B} = \mathcal{I}_B$. In particular, the logarithmic Jacobian ideal of order n of X_Γ does not depend on the generators of Γ .*

Proof. It is enough to show $\mathcal{I}_A = \mathcal{I}_{A \cup B}$. Lemma 3.13 states that $\mathcal{I}_A \subset \mathcal{I}_{A \cup B}$. Applying repeatedly lemma 3.14 we obtain the other inclusion. \square

Lemma 3.13. *With the notation of theorem 3.12, $\mathcal{I}_A \subset \mathcal{I}_{A \cup B}$.*

Proof. For $J \in S_A$, define $\bar{J} := \{(\beta, 0, \dots, 0) \in \mathbb{N}^{s+t} \mid \beta \in J\}$. The submatrix of $D_x^n(\varphi_{A \cup B})$ defined by \bar{J} is the same as the submatrix of $D_x^n(\varphi_A)$ defined by J . Therefore $\bar{J} \in S_{A \cup B}$. Thus, $X^{m_J} = X^{m_{\bar{J}}} \in \mathcal{I}_{A \cup B}$. \square

Lemma 3.14. *Let A be as in theorem 3.12 and $b \in \mathbb{N}A$. Let $A' = A \cup \{b\}$. Then $\mathcal{I}_{A'} \subset \mathcal{I}_A$.*

Proof. Consider the following partition of $S_{A'}$:

$$\begin{aligned} S_1 &:= \{\bar{J} \in S_{A'} \mid \beta(s+1) = 0 \text{ for all } \beta \in \bar{J}\}, \\ S_2 &:= \{\bar{J} \in S_{A'} \mid \beta(s+1) > 0 \text{ for some } \beta \in \bar{J}\}. \end{aligned}$$

By definition, $\mathcal{I}_{A'} = \langle \{X^{m_{\bar{J}}} \mid \bar{J} \in S_1\} \cup \{X^{m_{\bar{J}}} \mid \bar{J} \in S_2\} \rangle$. As in the proof of lemma 3.13, $\{X^{m_{\bar{J}}} \mid \bar{J} \in S_1\} \subset \mathcal{I}_A$. We claim that $\{X^{m_{\bar{J}}} \mid \bar{J} \in S_2\} \subset \langle \{X^{m_{\bar{J}}} \mid \bar{J} \in S_1\} \rangle$, implying the lemma. Now, to prove the claim we show that for $\bar{J} \in S_2$ there exists $J \in S_1$ and $\bar{\gamma} \in \Gamma$ such that $m_{\bar{J}} = m_J + \bar{\gamma}$. First, we need some notation.

- For $\gamma \leq \beta_i$, let $\epsilon_\gamma := (-1)^{|\beta_i - \gamma|} \binom{\beta_i}{\gamma}$. Then, by definition, $c_{\beta_i, \alpha_j} = \sum_{\gamma \leq \beta_i, \gamma \neq 0} \epsilon_\gamma \binom{A' \gamma}{\alpha_j}$ (see lemma 2.3).
- $c_{\beta_i} := \left(\sum_{\gamma \leq \beta_i, \gamma \neq 0} \epsilon_\gamma \binom{A' \gamma}{\alpha_j} \right)_{1 \leq j \leq \lambda_{d,n}}$ (c_{β_i} is the β_i -th row of $L_{\bar{J}}^c$).
- $v_\gamma := \left(\binom{A' \gamma}{\alpha_j} \right)_{1 \leq j \leq \lambda_{d,n}}$. Notice that by remark 4.3, $c_{\beta_i} = \sum_{\gamma \leq \beta_i, \gamma \neq 0} \epsilon_\gamma v_\gamma$ (that remark is stated for toric curves but it also holds for toric varieties of any dimension).

Let $\bar{J} = \{\beta_1, \dots, \beta_{\lambda_{d,n}}\} \in S_2$. Then $\det(L_{\bar{J}}^c) \neq 0$ and we can assume that $\beta_1(s+1) > 0$. Then the following holds:

1. There exists $\gamma' \leq \beta_1$ such that the matrix obtained by replacing the β_1 th row of $L_{\bar{J}}^c$ by $v_{\gamma'}$ has non-zero determinant.
2. There exists $\delta_0 \in \mathbb{N}^{s+1}$ such that $\delta_0(s+1) = 0$ and $A'\delta_0 = A'\gamma'$.
3. There exists $\delta \in \mathbb{N}^{s+1}$ such that $\delta \leq \delta_0$, $\delta(s+1) = 0$, and the matrix having as rows $c_\delta, c_{\beta_2}, \dots, c_{\beta_{\lambda_{d,n}}}$ has non-zero determinant.
4. Let $J_1 := \bar{J} \setminus \{\beta_1\} \cup \{\delta\}$. Then $J_1 \in S_{A'}$ and $m_{\bar{J}}$ equals m_{J_1} plus some element in Γ .

Notice that by applying 1 - 4 to any element of \bar{J} whose $(s+1)$ -entry is greater than zero, we obtain $J \in S_1$ and $\bar{\gamma} \in \Gamma$ with the desired properties. Now we prove the previous statements.

1. It follows immediately from:

$$\begin{aligned} 0 \neq \det(L_{\bar{J}}^c) &= \det \begin{pmatrix} c_{\beta_1} \\ c_{\beta_2} \\ \vdots \\ c_{\beta_{\lambda_{d,n}}} \end{pmatrix} = \det \begin{pmatrix} \sum_{\gamma \leq \beta_1, \gamma \neq 0} \epsilon_\gamma v_\gamma \\ c_{\beta_2} \\ \vdots \\ c_{\beta_{\lambda_{d,n}}} \end{pmatrix} \\ &= \sum_{\gamma \leq \beta_1, \gamma \neq 0} \epsilon_\gamma \det \begin{pmatrix} v_\gamma \\ c_{\beta_2} \\ \vdots \\ c_{\beta_{\lambda_{d,n}}} \end{pmatrix}. \end{aligned}$$

2. If $\gamma'(s+1) = 0$, let $\delta_0 := \gamma'$. Now suppose that $\gamma'(s+1) = k > 0$. Since $b \in \mathbb{N}A$, $b = \sum_{l=1}^s \lambda_l a_l$. Let $\delta_0(l) := \gamma'(l) + k\lambda_l$ for $l < s+1$ and $\delta_0(s+1) = 0$. Then

$$A'\delta_0 = \sum_{l=1}^s \delta_0(l) a_l = \sum_{l=1}^s (\gamma'(l) + k\lambda_l) a_l = \sum_{l=1}^s \gamma'(l) a_l + kb = A'\gamma'.$$

3. Let M denote the matrix whose rows are $c_{\delta_0}, c_{\beta_2}, \dots, c_{\beta_{\lambda_{d,n}}}$, in this order. If $\det(M) \neq 0$ let $\delta := \delta_0$. Suppose that $\det(M) = 0$. Then

$$0 = \det(M) = \sum_{\gamma < \delta_0, \gamma \neq 0} \epsilon_\gamma \det \begin{pmatrix} v_\gamma \\ c_{\beta_2} \\ \vdots \\ c_{\beta_{\lambda_{d,n}}} \end{pmatrix} + \det \begin{pmatrix} v_{\delta_0} \\ c_{\beta_2} \\ \vdots \\ c_{\beta_{\lambda_{d,n}}} \end{pmatrix}.$$

On the other hand, $A'\delta_0 = A'\gamma'$ implies $v_{\gamma'} = v_{\delta_0}$ and so

$$0 \neq \det \begin{pmatrix} v_{\gamma'} \\ c_{\beta_2} \\ \vdots \\ c_{\beta_{\lambda_{d,n}}} \end{pmatrix} = \det \begin{pmatrix} v_{\delta_0} \\ c_{\beta_2} \\ \vdots \\ c_{\beta_{\lambda_{d,n}}} \end{pmatrix}.$$

Therefore

$$0 \neq \sum_{\gamma < \delta_0, \gamma \neq 0} \epsilon_{\gamma} \det \begin{pmatrix} v_{\gamma} \\ c_{\beta_2} \\ \vdots \\ c_{\beta_{\lambda_{d,n}}} \end{pmatrix}.$$

Thus there exists $\delta_1 < \delta_0$ such that $\det(v_{\delta_1} c_{\beta_2} \cdots c_{\beta_{\lambda_{d,n}}}) \neq 0$. If $\det(c_{\delta_1} c_{\beta_2} \cdots c_{\beta_{\lambda_{d,n}}}) \neq 0$, let $\delta := \delta_1$. Otherwise repeat the previous process. This leads to a sequence $\delta_0 > \delta_1 > \cdots$. Since this sequence cannot decrease infinitely many times, we conclude that there exists $k \geq 0$ such that $\delta_0 > \delta_1 > \cdots > \delta_k =: \delta$ and $\det(c_{\delta} c_{\beta_2} \cdots c_{\beta_{\lambda_{d,n}}}) \neq 0$. In addition, since $\delta \leq \delta_0$ and $\delta_0(s+1) = 0$, we have $\delta(s+1) = 0$.

4. To show that $J_1 \in S_{A'}$ we only need to show that $|\delta| \leq n$ because we already know that $\det(L_{J_1}^c) \neq 0$. If $|\delta| > n$ then, by lemma 1.6 (ii), $c_{\delta} = 0$, which contradicts that $\det(L_{J_1}^c) \neq 0$. On the other hand, we know that $\delta \leq \delta_0$ and $\gamma' \leq \beta_1$. Let $\delta_0 = \delta + \delta'$ and $\beta_1 = \gamma' + \gamma''$. Then

$$A'\beta_1 = A'\gamma' + A'\gamma'' = A'\delta_0 + A'\gamma'' = A'\delta + A'\delta' + A'\gamma''.$$

This implies that $m_{\bar{J}}$ equals m_{J_1} plus an element from Γ .

□

4 Higher Nash blowup of toric curves

In this section we study in detail the higher Nash blowup of toric curves. In this section we use the following notation: $A = \{a_1, \dots, a_s\} \subset \mathbb{N}$, where $0 < a_1 < \dots < a_s$ and $\gcd(a_1, \dots, a_s) = 1$. Let $\Gamma = \mathbb{N}A \subset \mathbb{N}$. We assume that A is the minimal generating set of Γ . Let $X_{\Gamma} \subset \mathbb{K}^s$ be the corresponding toric curve.

According to theorem 3.9, $Nash_n(X_{\Gamma})$ is isomorphic to the blowup of the ideal \mathcal{I}_n . Since $\Gamma \subset \mathbb{R}_{\geq 0}$, it follows that the Newton polyhedron of \mathcal{I}_n

has only one vertex $m_{J_0} = \min\{m_J | J \in S_A\}$. In particular, $Nash_n(X_\Gamma)$ is determined by a single semigroup. We denote it as:

$$Nash_n(\Gamma) := \Gamma + \mathbb{N}(\{m_J - m_{J_0} | J \in S_A \setminus \{J_0\}\}).$$

Let us show how this semigroup looks like for $n = 1$. In this case, $S = \{e_1, \dots, e_s\}$, where the e'_i s denote the canonical basis of \mathbb{N}^s , $m_{e_i} = a_i$, and so $\min_i\{m_{e_i}\} = a_1$. Therefore

$$Nash_1(\Gamma) = \Gamma + \mathbb{N}(\{a_k - a_1 | k > 1\}).$$

Remark 4.1. The previous description is a particular case of the combinatorial description of the Nash blowup of toric varieties given in [11, 12] (see also [8], where the Nash blowup of toric curves is studied in detail).

We may ask the question: is there an explicit description for $Nash_n(\Gamma)$ as in $n = 1$? T. Yasuda made the following conjecture in a more general context.

Conjecture 4.2. [23, Conjecture 5.6] *Let X be a formal curve with associated semigroup $\Gamma = \{0 = s_0 < s_1 < \dots\}$. Let $Nash_n(\Gamma)$ be the associated semigroup of $Nash_n(X)$. Let $\Gamma^{(n)}$ be the semigroup generated by $s_m - s_l$, where $l \leq n < m$. Then $Nash_n(\Gamma) = \Gamma^{(n)}$.*

In what follows we prove that this conjecture is true for toric curves. However, in the final section we show that it fails in general.

In order to prove the conjecture in the toric case, first we need to study carefully some maximal minors of the higher-order Jacobian matrix. In section 2 we defined, for $J = \{\beta_1, \dots, \beta_n\} \subset \Lambda_{s,n}$, the matrix $L_J^c = (c_{\beta_i,j})_{i,j}$, where

$$c_{\beta_i,j} = \sum_{\gamma \leq \beta_i, \gamma \neq 0} (-1)^{|\beta_i - \gamma|} \binom{\beta_i}{\gamma} \binom{A \cdot \gamma}{j}.$$

Notice that in this case A is a vector in \mathbb{N}^s and $A \cdot \gamma$ is the usual dot product.

Remark 4.3. (a) For a fixed i , every entry of the i -th row of L_J^c has the same amount of summands and the same coefficients $(-1)^{|\beta_i - \gamma|} \binom{\beta_i}{\gamma}$. In other words, for a fixed row of L_J^c , the amount of summands and coefficients of its entries do not depend on j .

(b) Fix $i \in \{1, \dots, n\}$. We rewrite the sums $c_{\beta_i,j}$ as follows:

$$c_{\beta_i,j} = \binom{s_{i,1}}{j} + t_{i,2} \binom{s_{i,2}}{j} + \dots + t_{i,k_i} \binom{s_{i,k_i}}{j},$$

where $s_{i,1} := A \cdot \beta_i$, $s_{i,l} \in \{A \cdot \gamma \mid \gamma \leq \beta_i, 0 \neq \gamma \neq \beta_i\}$ for $1 < l \leq k_i$, and $t_{i,l} \in \mathbb{Z}$. Assume that $s_{i,1} > s_{i,2} > \dots > s_{i,k_i} > 0$. By (a), $\{s_{i,l}\}_l$, $\{t_{i,l}\}_l$ and k_i do not depend on j . Therefore, the i -th row of L_j^c can be written as:

$$\left(\binom{s_{i,1}}{1} + t_{i,2} \binom{s_{i,2}}{1} + \dots + t_{i,k_i} \binom{s_{i,k_i}}{1}, \dots, \binom{s_{i,1}}{n} + t_{i,2} \binom{s_{i,2}}{n} + \dots + t_{i,k_i} \binom{s_{i,k_i}}{n} \right).$$

Now we define some elementary operations on a matrix having the same shape as L_j^c . Given $k_i \in \mathbb{N}$ for $i \in \{1, \dots, n\}$, and $s_{i,l} \in \mathbb{N} \setminus \{0\}$, $t_{i,l} \in \mathbb{Q} \setminus \{0\}$ for $l \in \{1, \dots, k_i\}$, consider a matrix

$$D = \left(t_{i,1} \binom{s_{i,1}}{j} + t_{i,2} \binom{s_{i,2}}{j} + \dots + t_{i,k_i} \binom{s_{i,k_i}}{j} \right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}}.$$

Notice that if we fix i , the terms $k_i, s_{i,l}, t_{i,l}$ do not depend on j . Assume that $s_{i,1} > s_{i,2} > \dots > s_{i,k_i}$ for all $i \in \{1, \dots, n\}$. Finally, let R_i denote the i -th row of D .

Definition 4.4. We say that D satisfies (\star) if there exist $i, i' \in \{1, \dots, n\}$ such that $s_{i,1} = s_{i',1}$.

Using the following algorithm we show that, under some assumptions, we can perform elementary operations on the rows of D to obtain a matrix that does not satisfy the property (\star) .

Algorithm 4.5. Assume $\det D \neq 0$ and that D satisfies (\star) .

1. Replace the row R_i by $t_{i',1}R_i - t_{i,1}R_{i'}$.
2. Since $\det D \neq 0$ the new row cannot be the vector $\bar{0}$. Write this new vector as:

$$R'_i := \left(t'_{i,1} \binom{s'_{i,1}}{j} + t'_{i,2} \binom{s'_{i,2}}{j} + \dots + t'_{i,k'_i} \binom{s'_{i,k'_i}}{j} \right)_{1 \leq j \leq n},$$

where $t'_{i,l} \neq 0$ for all $l \in \{1, \dots, k'_i\}$ and $s'_{i,1} > \dots > s'_{i,k'_i}$. Notice that $s_{i,1} > s'_{i,1} > 0$.

3. Let

$$D' := \begin{pmatrix} R_1 \\ \vdots \\ R'_i \\ \vdots \\ R_n \end{pmatrix}.$$

- (i) If there exists $i'' \in \{1, \dots, n\} \setminus \{i\}$ such that $s'_{i,1} = s_{i'',1}$, then apply step 1 to R'_i . As before, we obtain a new element $s''_{i,1} \in \mathbb{N}$ such that $s_{i,1} > s'_{i,1} > s''_{i,1} > 0$.
- (ii) If there is no $i'' \in \{1, \dots, n\} \setminus \{i\}$ such that $s'_{i,1} = s_{i'',1}$ then stop.

Because of the decreasing sequence $s_{i,1} > s'_{i,1} > s''_{i,1} > \dots$, this algorithm must stop and it produces a new row that looks like

$$\left(u_{i,1} \binom{r_{i,1}}{j} + u_{i,2} \binom{r_{i,2}}{j} + \dots + u_{i,m_i} \binom{r_{i,m_i}}{j} \right)_{1 \leq j \leq n},$$

where $u_{i,l} \neq 0$ for all $l \in \{1, \dots, m_i\}$, $r_{i,1} > \dots > r_{i,m_i} > 0$ and $r_{i,1} \neq s_{i,1}$ for all $l \in \{1, \dots, n\} \setminus \{i\}$.

Applying this process every time that the new matrix satisfies (\star) , we finally get a matrix \bar{D}

$$\bar{D} = \left(u_{i,1} \binom{r_{i,1}}{j} + u_{i,2} \binom{r_{i,2}}{j} + \dots + u_{i,m_i} \binom{r_{i,m_i}}{j} \right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}},$$

such that $r_{i,1} > \dots > r_{i,m_i}$ for each i and $r_{i,1} \neq r_{i',1}$ for all $i \neq i'$.

Example 4.6. Consider the following matrix:

$$D = \begin{pmatrix} \binom{2}{1} & \binom{2}{2} & \binom{2}{3} \\ \binom{6}{1} - 2\binom{3}{1} & \binom{6}{2} - 2\binom{3}{2} & \binom{6}{3} - 2\binom{3}{3} \\ \binom{6}{1} - 3\binom{4}{1} + 3\binom{2}{1} & \binom{6}{2} - 3\binom{4}{2} + 3\binom{2}{2} & \binom{6}{3} - 3\binom{4}{3} + 3\binom{2}{3} \end{pmatrix}.$$

Notice that D satisfies (\star) . Applying algorithm 4.5 to the third row we obtain the matrix:

$$\bar{D} = \begin{pmatrix} \binom{2}{1} & \binom{2}{2} & \binom{2}{3} \\ \binom{6}{1} - 2\binom{3}{1} & \binom{6}{2} - 2\binom{3}{2} & \binom{6}{3} - 2\binom{3}{3} \\ -3\binom{4}{1} + 2\binom{3}{1} + 3\binom{2}{1} & -3\binom{4}{2} + 2\binom{3}{2} + 3\binom{2}{2} & -3\binom{4}{3} + 2\binom{3}{3} + 3\binom{2}{3} \end{pmatrix}.$$

4.1 A partial description of $Nash_n(X_\Gamma)$

The first step towards proving conjecture 4.2 for toric curves is to determine $\min_{J \in S_A} \{m_J\}$. Recall that for $J = \{\beta_1, \dots, \beta_n\} \subset \Lambda_{s,n}$, we defined $m_J = A \cdot \beta_1 + \dots + A \cdot \beta_n$. On the other hand, $A \cdot \beta_i \in \Gamma$ since A is the vector formed by the generators of Γ . Therefore, it is natural to expect that $\min_{J \in S_A} \{m_J\} = \sum_{i=1}^n s_i$. The goal of this subsection is to prove that this is indeed the case.

Proposition 4.7. *Let $J \subset \Lambda_{s,n}$, $|J| = n$, then $\min_{J \in S_A} \{m_J\} = \sum_{i=1}^n s_i$.*

Proof. This is proved in lemmas 4.9 and 4.13. \square

This proposition gives the following partial description of $Nash_n(\Gamma)$.

Corollary 4.8. $Nash_n(\Gamma) = \Gamma + \mathbb{N}(\{m_J - \sum_{i=1}^n s_i | J \in S_A\})$.

Lemma 4.9. *Let $J \in S_A$. Then $m_J \geq s_1 + \dots + s_n$. In particular, $\min_{J \in S_A} \{m_J\} \geq \sum_{i=1}^n s_i$.*

Proof. Let $J = \{\beta_1, \beta_2, \dots, \beta_n\}$. Using (b) of remark 4.3 we have

$$L_J^c = \left(\binom{s_{i,1}}{j} + t_{i,2} \binom{s_{i,2}}{j} + \dots + t_{i,k_i} \binom{s_{i,k_i}}{j} \right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}}$$

where $s_{i,l} \in \Gamma$ for each l , $s_{i,1} = A \cdot \beta_i$, and $s_{i,1} > s_{i,2} > \dots > s_{i,k_i}$. If $s_{i,1} \neq s_{i',1}$ for all $1 \leq i \neq i' \leq n$ then the statement follows.

Now suppose that there exist $\beta_i, \beta_{i'} \in J$, $i \neq i'$, such that $s_{i,1} = s_{i',1}$, i.e., L_J^c satisfies (\star) . Since $J \in S_A$, i.e., $\det(L_J^c) \neq 0$, we can apply algorithm 4.5 to obtain some elements $r_{1,1}, \dots, r_{n,1} \in \Gamma$ satisfying $r_{i,1} \neq r_{i',1}$ for all $i \neq i'$ and $s_{i,1} > r_{i,1}$ for some $i \in \{1, \dots, n\}$. Under these conditions we have

$$m_J = \sum_{i=1}^n A \cdot \beta_i = \sum_{i=1}^n s_{i,1} > \sum_{i=1}^n r_{i,1} \geq \sum_{i=1}^n s_i.$$

\square

To show that $\min_{J \in S_A} \{m_J\} = \sum_{i=1}^n s_i$ we need to show that, if $J = \{\beta_1, \dots, \beta_n\} \subset \Lambda_{s,n}$ is such that $A \cdot \beta_i = s_i$, then $J \in S_A$. In other words, we need to study J 's such that $\det(L_J^c) \neq 0$. We do not have a characterization of such J 's. However, in the following definition and lemma we give sufficient conditions for J to be in S_A .

Definition 4.10. Let $J \subset \mathbb{N}^s$ be a finite subset. We say that J satisfies (\star) if the following conditions hold:

- 1) For all $\beta, \beta' \in J$ such that $\beta \neq \beta'$, it holds $A \cdot \beta \neq A \cdot \beta'$.
- 2) For all $\beta \in J$ and $0 \neq \gamma < \beta$, there exists $\beta' \in J$ such that $A \cdot \gamma = A \cdot \beta'$.

Example 4.11. (i) Let $J = \{e_j, 2e_j, \dots, ne_j\}$, where e_j is a basic vector. Then J satisfies (\star) . Indeed, 1) follows by definition and 2) by the definition of $<$.

- (ii) Let $J = \{\beta_1, \beta_2, \dots, \beta_n\}$ be such that $A \cdot \beta_i = s_i$. Then J satisfies (*).
Indeed, 1) follows by definition and 2) follows from the fact that $\gamma < \beta$ implies $A \cdot \gamma < A \cdot \beta$.

Remark 4.12. Let $\beta \in \mathbb{N}^s$ be such that $|\beta| \geq n+1$. Then $A \cdot \beta > na_1 \geq s_n$. This implies that for $J = \{\beta_1, \dots, \beta_n\} \subset \mathbb{N}^s$ such that $A \cdot \beta_i = s_i$ for each i , we have $J \subset \Lambda_{s,n}$.

Lemma 4.13. Let $J \subset \Lambda_{s,n}$, $|J| = n$. If J satisfies (*) then $J \in S_A$. In particular, $\min_{J \in S_A} \{m_J\} \leq \sum_{i=1}^n s_i$.

Proof. The last statement follows from (ii) of example 4.11 and remark 4.12. Let $J = \{\beta_1, \dots, \beta_n\}$ satisfies (*). In particular, $A \cdot \beta_i \neq A \cdot \beta_{i'}$ for all $i \neq i'$. Assume $A \cdot \beta_1 < \dots < A \cdot \beta_n$. By (b) of remark 4.3, we can rewrite the matrix L_J^c as

$$\left(c_{\beta_i, j} \right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} = \left(\binom{s_{i,1}}{j} + t_{i,2} \binom{s_{i,2}}{j} + \dots + t_{i,k_i} \binom{s_{i,k_i}}{j} \right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}},$$

where $s_{i,1} = A \cdot \beta_i$, $s_{i,1} > s_{i,2} > \dots$, and $s_{i,l} = A \cdot \gamma$ for some $\gamma \leq \beta_i$.

Now we do some elementary operations on the n -th row of L_J^c :

$$\left(\binom{s_{n,1}}{j} + t_{n,2} \binom{s_{n,2}}{j} + \dots + t_{n,k_n} \binom{s_{n,k_n}}{j} \right)_{1 \leq j \leq n}.$$

We know that $s_{n,2} = A \cdot \gamma$ for some $\gamma < \beta_n$. Since J satisfies (*), there exists $\beta_{j_0} \in J$ such that $A \cdot \beta_{j_0} = A \cdot \gamma = s_{n,2}$. Then we subtract $t_{n,2}$ -times the row j_0 to the row n , thus obtaining

$$\left(\binom{s'_{n,1}}{j} + t'_{n,2} \binom{s'_{n,2}}{j} + \dots + t'_{n,k'_n} \binom{s'_{n,k'_n}}{j} \right)_{1 \leq j \leq n},$$

where $s'_{n,1} = s_{n,1}$, $s'_{n,2} > s'_{n,3} > \dots$, and $s_{n,2} > s'_{n,2}$. Now we have $s'_{n,2} = A \cdot \gamma'$ for some $\gamma' < \beta_n$ or some $\gamma' < \beta_{j_0}$. Once again, by (*) we can repeat the previous process to obtain a new element $s''_{n,2}$ such that $s_{n,2} > s'_{n,2} > s''_{n,2}$. Because of this decreasing sequence of natural numbers, the iteration of this process must stop turning the n -th row into

$$\left(\binom{s_{n,1}}{j} \right)_{1 \leq j \leq n}.$$

Applying this process to the other rows of L_J^c in an ascending way we obtain the matrix

$$\left(\binom{s_{i,1}}{j} \right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}}.$$

Notice that $s_{i,1} = A \cdot \beta_i \neq A \cdot \beta_{i'} = s_{i',1}$ for all $i \neq i'$. The following lemma shows that this matrix has non-zero determinant, thus concluding that $J \in S$. \square

Lemma 4.14. *Let $0 < c_1 < c_2 < \dots < c_n$ be natural numbers. Consider the matrix $L = \left(\binom{c_i}{j} \right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}}$. Then $\det L \neq 0$.*

Proof. For $j \in \{1, \dots, n\}$, consider the polynomial $b_j(x) = \frac{x(x-1)\dots(x-j+1)}{j!}$. Notice that if $x \in \mathbb{N}$, $b_j(x) = \binom{x}{j}$ and $\deg b_j(x) = j$. Thus

$$L = \left(\binom{c_i}{j} \right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} = \left(b_j(c_i) \right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}}.$$

Now we show that the columns of this matrix are linearly independent. Let $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ be such that $\sum_{j=1}^n \alpha_j b_j(c_i) = 0$, for each $i \in \{1, \dots, n\}$. Let $f(x) = \sum_{j=1}^n \alpha_j b_j(x)$. Then $\{c_1, \dots, c_n\}$ are roots of $f(x)$. But we also have $f(0) = 0$. Since $\deg f(x) \leq n$ it follows that $f(x) = 0$. As $\deg b_j(x) = j$ for each j , we conclude that $\alpha_j = 0$ for all j . In particular, $\det L \neq 0$. \square

4.2 Proof of conjecture 4.2 for toric curves and some consequences

Now we are ready to prove the main theorem of this section. Recall that by definition and corollary 4.8:

$$\begin{aligned} \Gamma^{(n)} &= \mathbb{N}(\{s_m - s_l | m > n, l \leq n\}), \\ \text{Nash}_n(\Gamma) &= \Gamma + \mathbb{N}(\{m_J - \sum_{l=1}^n s_l | J \in S_A\}). \end{aligned}$$

Theorem 4.15. $\Gamma^{(n)} = \text{Nash}_n(\Gamma)$.

Proof. This is proved in propositions 4.16 and 4.18. \square

Proposition 4.16. $\text{Nash}_n(\Gamma) \subset \Gamma^{(n)}$.

Proof. By corollary 4.8, it is enough to show that $a_i \in \Gamma^{(n)}$ for each $i \in \{1, \dots, s\}$ and $m_J - \sum_{l=1}^n s_l \in \Gamma^{(n)}$ for each $J \in S_A$.

We first prove $a_i \in \Gamma^{(n)}$. For $a_i \leq s_n$ there exists $m \in \mathbb{N}$ such that $ma_i \leq s_n < (m+1)a_i$. Then $a_i = (m+1)a_i - ma_i \in \Gamma^{(n)}$. If $a_i \geq s_n$ then $a_i + a_1 > s_n$, and $a_i = (a_i + a_1) - a_1 \in \Gamma^{(n)}$.

Now we prove that $m_J - \sum_{l=1}^n s_l \in \Gamma^{(n)}$ for each $J \in S_A$. Consider $J = \{\beta_1, \beta_2, \dots, \beta_n\} \in S_A$, let $s_{i,1} := A \cdot \beta_i$ and assume $s_{1,1} \leq \dots \leq s_{n,1}$. Let $k := \max\{l \in \{1, \dots, n\} \mid s_{l,1} \leq s_n\}$.

Case I: Suppose that $s_{1,1} < s_{2,1} < \dots < s_{k,1} \leq s_n$. Let $\psi = \{s_1, \dots, s_n\} \setminus \{s_{1,1}, \dots, s_{k,1}\}$. Write $\psi = \{r_{k+1}, \dots, r_n\}$. By definition of k and ψ we obtain:

$$m_J - \sum_{l=1}^n s_l = \sum_{l=1}^n s_{l,1} - \sum_{l=1}^n s_l = \sum_{l=k+1}^n s_{l,1} - \sum_{l=k+1}^n r_l \in \Gamma^{(n)}.$$

Case II: Suppose that there exist $i, i' \leq k$ such that $s_{i,1} = s_{i',1}$. We claim that for all $j \leq k$ there exist $r_{j,1} \in \Gamma$ such that $s_{j,1} - r_{j,1} \in \Gamma^{(n)}$ and $r_{j,1} \neq r_{j',1}$ for all $j \neq j'$. Assume this claim for the moment. For the elements $s_{m,1}$ with $m > k$, we have that $s_{m,1} > s_n$ and so $s_{m,1} - s_l \in \Gamma^{(n)}$ for any $l \leq n$. Let $\psi = \{s_1, \dots, s_n\} \setminus \{r_{1,1}, \dots, r_{k,1}\}$. Write $\psi = \{r_{k+1}, \dots, r_n\}$. As in the previous case, we conclude that

$$m_J - \sum_{l=1}^n s_l = \left(\sum_{l=1}^k s_{l,1} - \sum_{l=1}^k r_{l,1} \right) + \left(\sum_{l=k+1}^n s_{l,1} - \sum_{l=k+1}^n r_l \right) \in \Gamma^{(n)}.$$

Now we prove the claim. Since $J \in S_A$, we can apply algorithm 4.5 to any pair of rows of L_J^c , $i, i' \leq k$ such that $s_{i,1} = s_{i',1}$, to get a matrix

$$\bar{D} = \left(u_{i,1} \binom{r_{i,1}}{j} + u_{i,2} \binom{r_{i,2}}{j} + \dots + u_{i,m_i} \binom{r_{i,m_i}}{j} \right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}},$$

where $r_{i,1} \neq r_{i',1}$ for all $i, i' \leq k$. Let us show that $s_{i,1} - r_{i,1} \in \Gamma^{(n)}$ for all $i \in \{1, \dots, k\}$. We can assume that $s_{i,1} \neq r_{i,1}$.

In the first run of the algorithm we obtain an element $s'_{i,1} \in \Gamma$ such that $s_{i,1} > s'_{i,1} \geq r_{i,1}$ and $s'_{i,1} = A \cdot \gamma$ for some $\gamma < \beta_i$ or some $\gamma < \beta_{i'}$. This implies that $s_{i,1} - s'_{i,1} \in \Gamma$.

On the other hand, we know that $s'_{i,1} \geq r_{i,1}$ and $r_{i,1} \in \Gamma$. Therefore $s_{i,1} - s'_{i,1} + r_{i,1} \in \Gamma$ and $s_{i,1} - s'_{i,1} + r_{i,1} \leq s_n$. Consider the following set $\phi_i := \{s_l \in \Gamma \setminus \{0\} \mid s_l + r_{i,1} \leq s_n\}$. This set is not empty since $s_{i,1} - s'_{i,1} \in \phi_i$. Let $s_t := \max \phi_i$. If $s_{i,1} + s_t \leq s_n$, we have that $(s_{i,1} - s'_{i,1} + s_t) + r_{i,1} = s_{i,1} + s_t - (s'_{i,1} - r_{i,1}) \leq s_{i,1} + s_t \leq s_n$ and $s_{i,1} - s'_{i,1} + s_t > s_t$, which contradicts the maximality of s_t . Thus $s_{i,1} + s_t > s_n$ and

$$s_{i,1} - r_{i,1} = (s_{i,1} + s_t) - (s_t + r_{i,1}) \in \Gamma^{(n)}.$$

□

We need the following lemma to prove the remaining inclusion in theorem 4.15.

Lemma 4.17. *Let $s_m, s_i \in \Gamma$ be such that $m > n \geq i$ and $s_m - s_i \notin \Gamma$. Let $\beta_m \in \mathbb{N}^s$ be such that $A \cdot \beta_m = s_m$. Then there exists $\beta_0 \leq \beta_m$ such that $A \cdot \beta_0 > s_n$ and $|\beta_0| \leq n$.*

Proof. If $|\beta_m| \leq n$ then β_m satisfies the conditions of β_0 . Assume that $|\beta_m| > n$.

Suppose first that $a_2 \leq s_n$. The set $\{a_1, 2a_1, \dots, (n-1)a_1, a_2\}$ has n different elements of Γ implying $na_1 > s_n$. Let β be such that $|\beta| = n$. Then $A \cdot \beta \geq ns_1 = na_1 > s_n$. In particular, any $\beta \leq \beta_m$ such that $|\beta| = n$ satisfies the conditions on the lemma.

Now suppose $s_n < a_2$. Then $s_k = ks_1$ for all $k \leq n$. Notice that if $\beta_m(j) = 0$ for all $j > 1$, then $s_m - s_i = |\beta_m|a_1 - ia_1 \in \Gamma$ which contradicts the hypothesis. Thus there exists $j > 1$ such that $\beta_m(j) \neq 0$. Consider $\beta_0 = e_j$, then $A \cdot \beta_0 = A \cdot e_j = a_j \geq a_2 > s_n$ and $|\beta_0| = 1 \leq n$. \square

Proposition 4.18. $\Gamma^{(n)} \subset \text{Nash}_n(\Gamma)$.

Proof. Throughout this proof we fix $m, i \in \mathbb{N}$ such that $m > n \geq i$. Let $s_m - s_i \in \Gamma^{(n)}$.

Case I: Suppose that $s_m - s_i \in \Gamma$. Then $s_m - s_i \in \text{Nash}_n(\Gamma)$ by definition.

Case II: Suppose that $s_m - s_i \notin \Gamma$. Fix $\beta_m \in \mathbb{N}^s$ such that $A \cdot \beta_m = s_m$.

We claim that there exist $\beta_0 \in \mathbb{N}^s$, $J_0 = \{\beta_1, \dots, \beta_n\} \subset \mathbb{N}^s$, and $i \leq l \leq n$ such that:

- (1) $\beta_0 \leq \beta_m$.
- (2) $A \cdot \beta_j = s_j$ for each $j \in \{1, \dots, n\}$.
- (3) $s_l - s_i \in \Gamma$.
- (4) $J := (J_0 \setminus \{\beta_l\}) \cup \{\beta_0\}$ satisfies (*). In particular,

$$A \cdot \beta_0 - s_l = m_J - \sum_{i=1}^n s_i \in \text{Nash}_n(\Gamma).$$

Assume this claim for the moment. Let $\delta \in \mathbb{N}^s$ be such that $\beta_m = \beta_0 + \delta$. In particular, $s_m = A \cdot \beta_m = A \cdot \beta_0 + A \cdot \delta$. Then, since $A \cdot \delta \in \Gamma$, we conclude

$$s_m - s_i = (A \cdot \beta_0 - s_l) + A \cdot \delta + (s_l - s_i) \in \text{Nash}_n(\Gamma).$$

Now we prove the claim. We first show that there is a $\beta_0 \in \mathbb{N}^s$ satisfying (1) and some extra conditions needed for the proof of (4). Let $T := \{\gamma \in \mathbb{N}^s \mid \gamma \leq \beta_m\}$. We write this set as $T = T_{\leq} \sqcup T_{>}$, where

$$\begin{aligned} T_{\leq} &:= \{\gamma \in T \mid A \cdot \gamma \leq s_n\}, \\ T_{>} &:= \{\gamma \in T \mid A \cdot \gamma > s_n\}. \end{aligned}$$

Notice that $\beta_m \in T_{>}$. Let $\beta_0 \leq \beta_m$ be a minimal element in $T_{>}$ such that $\beta_0 \in \Lambda_{s,n}$ (such an element exists by lemma 4.17). By construction, β_0 has the following properties:

- a) For all $\gamma < \beta_0$, $\gamma \in T_{\leq}$.
- b) For all $\bar{\beta}_i$ such that $A \cdot \bar{\beta}_i = s_i$ it holds $\beta_0 \not\preceq \bar{\beta}_i$ (this is true because $\beta_m \geq \beta_0$ and $s_m - s_i \notin \Gamma$ implies that $\beta_m \not\preceq \bar{\beta}_i$).

Now we prove the existence of $J_0 = \{\beta_1, \dots, \beta_n\} \subset \mathbb{N}^s$, and $i \leq l \leq n$ satisfying (2) and (3) and some extra conditions needed for the proof of (4). Define the set (recall that i is fixed):

$$\Omega := \{s_j \in \{s_1, \dots, s_n\} \mid \forall \beta' \text{ such that } A \cdot \beta' = s_j, \exists \gamma \leq \beta' \text{ such that } A \cdot \gamma = s_i\}.$$

If $s_j \in \{s_1, \dots, s_n\} \setminus \Omega$, consider $\beta_j \in \mathbb{N}^s$ such that $A \cdot \beta_j = s_j$ and for all $\gamma \leq \beta_j$, $A \cdot \gamma \neq s_i$. If $s_j \in \Omega$, consider any $\beta_j \in \mathbb{N}^s$ such that $A \cdot \beta_j = s_j$. Let $J_0 := \{\beta_1, \dots, \beta_n\} \subset \mathbb{N}^s$. By remark 4.12, we have that $J_0 \subset \Lambda_{s,n}$. Notice that $\Omega \neq \emptyset$, since $s_i \in \Omega$. Let $s_l := \max\{\Omega\}$. In particular, $s_l \in \Omega$ and so $s_l - s_i \in \Gamma$.

It remains to prove that $J := (J_0 \setminus \{\beta_l\}) \cup \{\beta_0\}$ satisfies (*) (see definition 4.10). By construction and since $A \cdot \beta_0 > s_n$, we have 1) in the definition of (*).

Now let $\beta_k \in J_0 \setminus \{\beta_l\}$. We want to show that if $\gamma < \beta_k$ then there exists $\beta' \in J$ such that $A \cdot \beta' = A \cdot \gamma$. If $k < l$ this condition is satisfied (see example 4.11). Suppose $k > l$ (in particular, $s_k \notin \Omega$). If $\gamma < \beta_k$ is such that $A \cdot \gamma = s_j \neq s_l$ then by making $\beta' = \beta_j$ the condition is satisfied. Suppose that $A \cdot \gamma = s_l$. Since $s_l \in \Omega$ there exists $\gamma' \leq \gamma$ such that $A \cdot \gamma' = s_i$. Since $\gamma < \beta_k$, it follows that $\gamma' < \beta_k$. This is a contradiction since β_k was chosen so that for all $\delta < \beta_k$ we have $A \cdot \delta \neq s_i$. Therefore, for all $\gamma \leq \beta_k$, $A \cdot \gamma \neq s_l$. This shows that every element of $J_0 \setminus \{\beta_l\}$ satisfies 2) in the definition of (*).

Now consider $\gamma < \beta_0$. By property a) above, we have that $A \cdot \gamma \leq s_n$. If $\gamma < \beta_0$ is such that $A \cdot \gamma = s_j \neq s_l$ then, as before, by making $\beta' = \beta_j$ the condition is satisfied. Suppose that $A \cdot \gamma = s_l$. As before, there exists $\gamma' \leq \gamma$

such that $A \cdot \gamma' = s_i$. Since $\gamma < \beta_0$, it follows that $\gamma' < \beta_0$. This contradicts property b) above. We conclude that J satisfies (*). \square

Theorem 4.15 has two immediate consequences. The first one is about solving toric curves by applying once the higher Nash blowup for n sufficiently large. This gives a combinatorial proof of Yasuda's theorem on one-step resolution of curves by higher Nash blowups in the case of toric curves. The second result is the analogous of Nobile's theorem for the higher Nash blowup of toric curves.

Corollary 4.19. *$Nash_n(X_\Gamma)$ is non-singular if and only if $s_n + 1 \in \Gamma$. In particular, $Nash_n(X_\Gamma)$ is non-singular for $n \gg 0$.*

Proof. Notice that for all $m > n$ and $i \leq n$, we have that $s_{n+1} + s_i \leq s_m + s_n$, then $s_{n+1} - s_n \leq s_m - s_i$. Thus, $s_{n+1} - s_n = \min\{\Gamma^{(n)} \setminus \{0\}\} = \min\{Nash_n(\Gamma) \setminus \{0\}\}$. Then $Nash_n(X_\Gamma)$ is non-singular if and only if $Nash_n(\Gamma) = \mathbb{N}(\{1\})$ if and only if $1 = s_{n+1} - s_n$ if and only if $s_n + 1 = s_{n+1} \in \Gamma$. \square

Corollary 4.20. *$Nash_n(X_\Gamma) \cong X_\Gamma$ if and only if X_Γ is non-singular.*

Proof. Suppose that X_Γ is singular, i.e., $1 < a_1$. We are going to show that $\Gamma \subsetneq Nash_n(\Gamma)$, which implies $Nash_n(X_\Gamma) \not\cong X_\Gamma$.

Let $a_2 = qa_1 + r$, where $0 < r < a_1$ and $q \geq 1$. With this notation we have $s_1 = a_1, \dots, s_q = qa_1, s_{q+1} = a_2$. If $n \leq q$ then $s_n \leq s_q = qa_1 < a_2$ and so $a_2 - a_1 \in \Gamma^{(n)} = Nash_n(\Gamma)$. But we also have $a_2 - a_1 = (q-1)a_1 + r \notin \Gamma$.

Suppose that $n > q$. Consider the following subset of Γ :

$$\{s_{q+1} = qa_1 + r, (q+1)a_1, (q+1)a_1 + r, (q+2)a_1, (q+2)a_1 + r, \dots\}.$$

The elements on this subset are not necessarily consecutive elements in Γ . Therefore, for $p > q$ it follows $s_{p+1} - s_p \leq \max\{a_1 - r, r\} < a_1$. In particular, $s_{n+1} - s_n < a_1$. Thus, $s_{n+1} - s_n \in Nash_n(\Gamma)$ but $s_{n+1} - s_n \notin \Gamma$. \square

5 Counterexample to the conjecture

In section 4 we stated and proved a conjecture by T. Yasuda for toric curves. In this section we exhibit a family of non-monomial curves showing that the conjecture is false in general.

Example 5.1. Consider the plane curve $C \subset \mathbb{C}^2$ parametrized by

$$t \mapsto (t^4, t^6 + t^7).$$

The associated semigroup of C is $\Gamma = \{0, 4, 6, 8, 10, 12, 13, 14, m | m \geq 16\}$. Yasuda's conjecture states that the semigroup of $Nash_1(C)$ is $\Gamma^{(1)} = \mathbb{N}(2, 9)$. However, the Nash blowup of order 1 of C is parametrized by

$$t \mapsto \left(t^4, t^6 + t^7, \frac{6}{4}t^2 + \frac{7}{4}t^3 \right).$$

Using the first and third terms of the parametrization we obtain $Nash_1(\Gamma) = \mathbb{N}(2, 5)$. We conclude that $Nash_1(\Gamma) \neq \Gamma^{(1)}$.

We may still ask whether the conjecture holds for $n \gg 0$. In what follows we construct a family of plane curves $\{C_n\}_{n \geq 1}$, with numerical semigroup Γ_n , such that $Nash_n(\Gamma_n) \neq (\Gamma_n)^{(n)}$.

Fix $n \geq 1$. Consider the plane curve C_n with parametrization

$$\varphi(t) = (t^4, t^{4n+2} + t^{4n+3}).$$

Let Γ_n be the corresponding semigroup. Notice that the first n non-zero terms of Γ_n is the set $\{4, 8, \dots, 4n\}$. In addition, the first odd number that appears in Γ_n is $8n+5$ (it appears as the order of $(t^{4n+2} + t^{4n+3})^2 - (t^4)^{2n+1}$). In particular, the first odd number that appears in $(\Gamma_n)^{(n)}$ is $8n+5-4n = 4n+5$. We claim that $5 \in Nash_n(\Gamma_n)$ implying that $Nash_n(\Gamma_n) \neq (\Gamma_n)^{(n)}$.

To prove the claim we need to compute some maximal minors of the matrix

$$\left(\frac{1}{\alpha!} \frac{\partial^\alpha (\varphi - \varphi(t))^\beta}{\partial T^\alpha} \Big|_t \right)_{\beta \in \Lambda_{2,n}, \alpha \in \Lambda_{1,n}}$$

Let $J_1 = \{e_1, 2e_1, \dots, ne_1\}$ and $J_2 = \{e_1, 2e_1, \dots, (n-1)e_1, e_2\}$. We first show that the minors of the submatrices defined by J_1 and J_2 are not zero.

Let L_{J_1} be the submatrix defined by J_1 . Notice that the rows of L_{J_1} only consider the first term of $\varphi(t)$, which is a monomial. Therefore, by example 4.11 and lemma 4.13, $\det L_{J_1} \neq 0$. In addition, by proposition 2.4, $\det L_{J_1} = c \cdot t^{\sum_{k=1}^n 4k-k} = c \cdot t^{\frac{3n(n+1)}{2}}$, with c a non-zero constant.

Now, for J_2 , notice that the first $n-1$ rows of L_{J_2} only consider the monomial term of $\varphi(t)$. Using lemma 2.3 we obtain that the (i, j) -entry of L_{J_2} is $c_{ie_1, j} t^{4i-j}$, for $1 \leq i < n$ and $1 \leq j \leq n$. On the other hand, the n th row of L_{J_2} can be described as follows. Since $|e_2| = 1$, by lemma 1.6 we obtain

$$\frac{1}{j!} \frac{\partial^j (\varphi - \varphi(t))^{e_2}}{\partial T^j} \Big|_t = \binom{4n+2}{j} t^{4n+2-j} + \binom{4n+3}{j} t^{4n+3-j}.$$

Summarizing, the matrix L_{J_2} is:

$$\begin{pmatrix} c_{e_1,1}t^{4-1} & \cdots & c_{e_1,n}t^{4-n} \\ \vdots & & \vdots \\ c_{(n-1)e_1,1}t^{4(n-1)-1} & \cdots & c_{(n-1)e_1,n}t^{4(n-1)-n} \\ \binom{4n+2}{1}t^{4n+2-1} + \binom{4n+3}{1}t^{4n+3-1} & \cdots & \binom{4n+2}{n}t^{4n+2-n} + \binom{4n+3}{n}t^{4n+3-n} \end{pmatrix}.$$

Multiply the j th column by t^j . Then, for $1 \leq i < n$ multiply the i th row by t^{-4i} . Finally, multiply the n th row by t^{-4n-2} to obtain

$$\det L_{J_2} = \left(t^{\frac{3n(n+1)}{2}+2}\right) \det \begin{pmatrix} c_{e_1,1} & \cdots & c_{e_1,n} \\ \vdots & & \vdots \\ \binom{4n+2}{1} + \binom{4n+3}{1}t & \cdots & \binom{4n+2}{n} + \binom{4n+3}{n}t \end{pmatrix}.$$

Applying the same method of the proof of proposition 4.13 in the first $n-1$ rows and using basic properties of determinants, we get that

$$\begin{aligned} \det L_{J_2} = & t^{\frac{3n(n+1)}{2}+2} \det \begin{pmatrix} \binom{4}{1} & \cdots & \binom{4}{n} \\ \vdots & & \vdots \\ \binom{4(n-1)}{1} & \cdots & \binom{4(n-1)}{n} \\ \binom{4n+2}{1} & \cdots & \binom{4n+2}{n} \end{pmatrix} \\ & + t^{\frac{3n(n+1)}{2}+3} \det \begin{pmatrix} \binom{4}{1} & \cdots & \binom{4}{n} \\ \vdots & & \vdots \\ \binom{4(n-1)}{1} & \cdots & \binom{4(n-1)}{n} \\ \binom{4n+3}{1} & \cdots & \binom{4n+3}{n} \end{pmatrix}. \end{aligned}$$

The determinants appearing in the sum are non-zero by lemma 4.14. Therefore $\det L_{J_2} \neq 0$.

Now we need to prove that $\det L_{J_1}$ has the minimum order over all non-zero minors of the higher-order Jacobian matrix of φ .

Consider $\beta = (a_1, a_2) \in \mathbb{N}^2$ and suppose that $a_2 > 0$. Notice that if the polynomial

$$\frac{1}{m!} \frac{\partial^m (T^4 - t^4)^{a_1} (T^{4n+2} + T^{4n+3} - t^{4n+2} - t^{4n+3})^{a_2}}{\partial T^m} \Big|_t$$

is non-zero, then its order is greater or equal than $4n+2-m$. Let $J = \{\beta_1, \dots, \beta_n\} \subset \Lambda_{2,n}$ be such that $J \neq \{e_1, \dots, ne_1\}$. In particular, the second entry of β_i is non-zero, for some i . Reorder J in such a way that

$\beta_i(2) \neq 0$ for $1 \leq i \leq k$ and $\beta_j(2) = 0$ for $k < j \leq n$. Then, if $j > k$, $\beta_j = m_j e_1$ with $1 \leq m_j \leq n$ and if $j > i > k$, $m_j \neq m_i$.

Let us show that if $\det L_J \neq 0$ then $\text{ord}(\det(L_J)) > \frac{3n(n+1)}{2}$. To begin with,

$$\det L_J = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^n a_{i,\sigma(i)} = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^n \frac{1}{\sigma(i)!} \frac{\partial^{\sigma(i)}(\varphi - \varphi(t))^{\beta_i}}{\partial T^{\sigma(i)}} \Big|_t,$$

where S_n is the symmetric group. Let $A_\sigma = \prod_{i=1}^n \frac{1}{\sigma(i)!} \frac{\partial^{\sigma(i)}(\varphi - \varphi(t))^{\beta_i}}{\partial T^{\sigma(i)}} \Big|_t$. The claim follows if we can prove that, for all σ such that $A_\sigma \neq 0$, $\text{ord}(A_\sigma) > \frac{3n(n+1)}{2}$. But this is true since:

$$\begin{aligned} \text{ord}(A_\sigma) &= \sum_{i=1}^n \text{ord}\left(\frac{\partial^{\sigma(i)}(\varphi - \varphi(t))^{\beta_i}}{\partial T^{\sigma(i)}} \Big|_t\right) \\ &\geq \sum_{i=1}^k (4n + 2 - \sigma(i)) + \sum_{j=k+1}^n (4m_j - \sigma(j)) \\ &= 2k + 4(nk + \sum_{j=k+1}^n m_j) - \frac{n(n+1)}{2} \geq 2k + 4 \sum_{j=1}^n j - \frac{n(n+1)}{2} \\ &= 2k + \frac{3n(n+1)}{2} > \frac{3n(n+1)}{2}. \end{aligned}$$

Using all previous claims we see that the Plücker coordinates of $T_{\varphi(t)}^n C_n$, for $t \neq 0$, look like:

$$(\dots : ct^{\frac{3n(n+1)}{2}} : c_1 t^{\frac{3n(n+1)}{2}+2} + c_2 t^{\frac{3n(n+1)}{2}+3} : \dots),$$

with c, c_1, c_2 non-zero constants. Since the coordinate defined by J_1 has the minimum order, $\text{Nash}_n(C_n) \subset U_{J_1}$, i.e., the affine chart obtained from dividing over $ct^{\frac{3n(n+1)}{2}}$. In particular, the parametrization of $\text{Nash}_n(C_n)$ has the term

$$\frac{c_1}{c} t^2 + \frac{c_2}{c} t^3.$$

Now proceed as in example 5.1 to show that $5 \in \text{Nash}_n(\Gamma_n)$.

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