

Bayesian entropy estimation applied to non-Gaussian robust image segmentation

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Abstract—We introduce a new approach for robust image segmentation combining two strategies within a Bayesian framework. The first one is to use a Markov random field (MRF) which allows to introduce prior information with the purpose of image edges preservation. The second strategy comes from the fact that the probability density function (pdf) of the likelihood function is non-Gaussian or unknown, so it should be approximated by an estimated version, which is obtained by using the classical non-parametric or kernel density estimation. This lead us to the definition of a new maximum a posteriori (MAP) estimator based on the minimization of the entropy of the estimated pdf of the likelihood function and the MRF at the same time, named MAP entropy estimator (MAPEE). Some experiments were made for different kind of images degraded with impulsive noise (salt & pepper) and the segmentation results are very satisfactory and promising.

Index Terms—Robust image segmentation, Markov random fields, Bayesian estimation, non-parametric density estimation, entropy minimization.

1. INTRODUCTION

Image segmentation is one of the most important tasks in image processing. Nevertheless, digital images are usually degraded by blurring or noise, resulting in degraded or distorted images and producing, as a consequence, inadequate segmentation results. A degradation process, Fig. 1, can be described as a degradation function H that, together with an additive noise term n , it operates on an input image x and produces a degraded image y , it means

$$y = Hx + n. \quad (1)$$

An approach that have helped significantly to solve the problem of segmentation of degraded images is the use of Markov random fields (MRF) within a Bayesian framework [1-5]. This is because MRFs enables posing this problem, and many others in image processing, as statistical estimation problems [4] where the solution is going to be estimated from the degraded image. The basic premise is that neighborhood pixels are expected to have similar characteristics [5, 6].

Usually, information provided by the input image is not enough for an accurate estimation of the original image, so *a priori* information (assumptions about the structure of x) need to be introduced in the estimation process [7]. The *a priori* knowledge is given in terms of a probability distribution, that

together with a probabilistic description of the noise that corrupts the observations, allows the use of Bayes theory to compute the *posterior* distribution which represents the likelihood of a solution x given the observations y [6, 8].

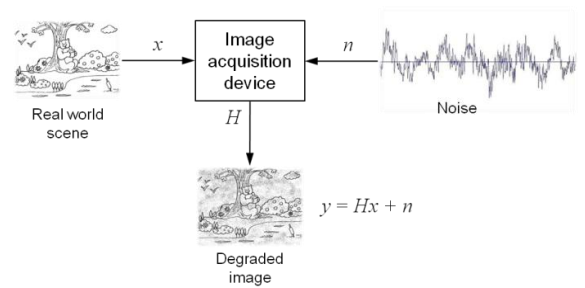


Fig. 1. Degradation process of an image.

The basic idea in Bayesian estimation is to construct a Maximum A Posteriori (MAP) by using MRFs. In the case of classical MAP filters, usually the additive Gaussian noise is considered, however in some applications this noise is non-Gaussian or unknown [9]. This becomes in a new source of information which imposes additional constraints in the image processing context (the spatial information) that represents the likelihood function or correlation between the intensity values of a well specified neighborhood of pixels.

The Bayes rule states that:

$$p(x|y) = \frac{p(y|x)p(x)}{p(y)}, \quad (2)$$

where $p(x)$ corresponds to a probabilistic description of the real world, that we are trying to estimate, before collecting data; $p(y|x)$ is a description of the behavior of noise that relates the original state x to the sampled input image or sensor values y ; $p(x|y)$ is a probabilistic description of the current estimation of the original scene x , given the observed data y [10]; $p(y)$ is the density function of y and is constant if the observed image is provided [6, 11].

The MAP estimator is defined by:

$$\begin{aligned} \hat{x}_{\text{MAP}} &= \underset{x \in \mathbb{X}}{\operatorname{argmax}} \{p(x|y)\} \\ &= \underset{x \in \mathbb{X}}{\operatorname{argmax}} \{\log p(y|x) + \log g(x)\} \\ &= \underset{x \in \mathbb{X}}{\operatorname{argmin}} \{-\log p(y|x) - \log g(x)\}, \end{aligned} \quad (3)$$

where $g(x)$ is a MRF function that models, as a probability distribution, the prior information of the phenomenon to be estimated, \mathbb{X} is the set of pixels capable to maximize $p(x|y)$ and $p(y|x)$ is the likelihood function from y given x [12].

In this work we take as a starting point the use of a recent MRF model, named semi-Huber [6], in the second term of equation (3), because of its simplicity and minor number of parameters. The new approach of entropy estimation is going to be present in the variation of the first term of that expression. In [6], as in some other works [1, 2, 12], the first term in the MAP estimator was defined as a quadratic function of the differences between real and observed data, because of the additive noise regarded was Gaussian. For the experiments performed here, it is considered impulsive noise (*salt & pepper*) which does not follow a specific pattern and is a more degrading kind of noise.

Modeling in this new context lead us to assume a limited knowledge about the image noise pdf, so it is proposed to use the data itself to obtain a non-parametric Entropy Estimate (EE) of the log-likelihood pdf [13-15]. Then it will be optimized together with the log-MRF to obtain the MAP image segmentation. The rest of the paper is as follows: section 2 describes the definition of the approximation of the log-likelihood function by entropy estimation. Section 3 gives a brief background about the kernel structure for the density estimation function and the complete definition of the new MAP entropy estimator is deduced. In section 4 some experiments and results are presented and discussed. Finally in section 5 some concluding comments are given.

2. LOG-LIKELIHOOD APPROXIMATED BY EE.

A. The general problem of regression.

A wide variety of applications in signal processing and instrumentation are based on statistical modeling analysis. The linear regression model is one of the most used

$$y_{i,j} = x_{i,j}^T \theta_{i,j} + e_{i,j}, \quad \text{con } e \sim p(e), \quad (4)$$

where y represents the response to x explicative variables for $i = 1, \dots, N$ and $j = 1, \dots, M$, and to a system parameterized by θ , a set of functional parameters associated to the data (y, x) , which will be estimated by an identification procedure. The e variables are the errors, that model the system as a set of random processes which are independent and identically distributed accordingly to $p(e)$.

A natural extension of the linear regression model is the non-linear regression model, but now it is based on a parameterized function $f(\cdot)$

$$y_{i,j} = f(x, \theta)_{i,j} + e_{i,j}, \quad \text{con } e \sim p(e). \quad (5)$$

This function is nonlinear with respect to the parameters, and its use is also considered because it has been shown in a large variety of signal processing and control applications that modeling when using nonlinear functions could be more realistic [15].

There exist some classical techniques for the estimation of

θ , for example Least Squares (LS), Maximum Likelihood (ML), among others. In this work it is proposed a MAP estimation based on the entropy minimization of an estimated version of the density of the errors ($\hat{p}_{n,h}(e)$).

B. Likelihood pdf entropy estimators (EE).

A classical procedure to estimate x when θ is known, is based in a cost function or criterion $\mathcal{J}(x)$ which varies in function $\psi(\cdot)$ of the residuals or noise $e(x)$, where

$$e_{i,j}(x) = y_{i,j} - f(x, \theta)_{i,j}. \quad (6)$$

Thus,

$$\mathcal{J}(x) = \sum_{i=1}^N \sum_{j=1}^M \psi(e_{i,j}(x)). \quad (7)$$

Our proposition for a new MAP scheme is to use the semi-Huber MRF introduced in [6], together with a kernel estimator taken from [13-15] to obtain cost functions or criterions based on the entropy of the approximated likelihood function $\hat{p}_{n,h}(e)$. Thus, $-\log p(y|x)$ is built on the basis of the entropy of an estimated version $\hat{p}_{n,h}(e)$ of the distribution $p(e)$. A first proposition is due to Pronzato and Thierry [16, 17], where the approximation is obtained using the classical kernel estimators which use the empirical distribution of the random vector $e_{1,1}(x), \dots, e_{n,n}(x)$:

$$\begin{aligned} \hat{p}_{n,h}(e) &= \hat{p}_{n,h}(e|e_{1,1}(x), \dots, e_{n,n}(x)) \\ &= \frac{1}{n^2} \sum_{k=1}^n \sum_{l=1}^n K_h(e - e_{k,l}). \end{aligned} \quad (8)$$

$K(\cdot)$ is a kernel weighted function which satisfies some imposed conditions treated in the work of Masry [18] and subsequently taken back by Devroye [19-21], Berline [22], and Loader [23] in some of their research work. The bandwidth $h = h_n$ is given in function of the sample size and can be considered as a sequence of positive numbers that must satisfy $h_n \rightarrow 0$ and $nh_n \rightarrow \infty$ when $n \rightarrow \infty$. The strong uniform consistency of $\hat{p}_{n,h}(e)$ and its convergence toward $p(e)$, depend on a convenient procedure of bandwidth selection [14]. A simple and faster procedure is the technique proposed and developed by Terrell [24, 25].

Assuming that $\hat{p}_{n,h}(e)$ converges and is consistent, such that $\hat{p}_{n,h}(e) \rightarrow p(e)$, then the entropy criterion over $\hat{p}_{n,h}(e)$ can be approximated to $-\log p(y|x)$. The fact that the entropy of any probability density function is invariant by translation, leads to consider one practical artifact to build an extended criterion based on the residuals or noise extended vector, given by:

$$e_E = \{e_{1,1}(x), \dots, e_{n,n}(x), -e_{1,1}(x), \dots, -e_{n,n}(x)\}, \quad (9)$$

and on a suitable choice of h :

$$\mathcal{J}_e(x) = H_A(\hat{p}_{n,h}(e_E)) \approx -\log p(y|x), \quad (10)$$

where

$$H_A(f) = - \int_{-A_n}^{A_n} f(x) \log f(x) dx. \quad (11)$$

Then a first version of the MAP Entropy Estimator (MAPEE) assuming unknown noise pdf, can be constructed:

$$\hat{x}_{\text{MAPEE}} = \arg \min_{x \in \mathbb{X}} \left\{ H_A(\hat{p}_{n,h}(e_E)) - \log g(x) \right\}. \quad (12)$$

3. THE MAP ENTROPY ESTIMATOR (MAPEE).

A function of the form $K(z)$ is assumed as a fixed kernel

$$K_h(z) = \frac{1}{h^d} K\left(\frac{z}{h}\right), \quad (13)$$

where $h > 0$ is a parameter called the kernel bandwidth. The fundamental problem in kernel density estimation lies in both the selection of an appropriate value for h and the selection of the kernel structure. Taking as a reference the works [13-15], the Hilbert kernels [20] was selected here, this is because of the results presented in the referred papers and mainly because of their structure is such that they avoid the bandwidth selection and their performance depend on other parameters, which selection is very easy.

A. The Hilbert kernel.

Equation (13) is considered equivalent to $K(u) = 1/\|u\|^d$, where the smoothing factor h is canceled, obtaining:

$$\hat{p}_n(z) = \frac{1}{n^2} \sum_{k=1}^n \sum_{l=1}^n \frac{1}{\|z - z_{k,l}\|^d}. \quad (14)$$

The consistency of this class of estimators is proved in [20]. The Hilbert density estimate of order k ($k > 0$) is a redefined subclass that avoids the infinite peaks produced during estimation; in one dimensional case and using the value of $k = 2$ the kernel estimate is given by:

$$\hat{p}_n(z) = \sqrt{\frac{4}{V_d^2 \pi n(n-1) \log n}} \sum_{1 \leq i < j \leq n} \frac{1}{\text{Den}_{i,j}}, \quad (15)$$

where $\text{Den}_{i,j} = \|z - z_i\|^{2d} + \|z - z_j\|^{2d}$, V_d is the volume of the unit ball in \mathbb{R}^b and $\|\cdot\|$ denotes the L_2 metric on \mathbb{R}^d . Finally, it is assumed that $\hat{p}_n(z) \rightarrow p(z)$ at least in probability for almost all z . For a suitable choice of d and k , this estimator could be ‘‘blind asymptotically efficient’’.

B. The semi-Hubber MRF.

In this section it is obtained the complete cost function structure for the named \hat{x}_{MAPEE} estimator derived from (3). The first term has been already described in the previous section, corresponding to the new approach proposed here; and the second one ($\log g(x)$) is based on the semi-Huber MRF introduced in [6, 26].

The Huber-like norm or semi-Huber potential function, for the two dimensional case, is given by:

$$\log g(x) = -\lambda \left(\sum_{\{s,r\} \in \mathbb{C}} b_{sr} \rho_1(x) \right) + c, \quad (16)$$

where s is the site or pixel of interest, r corresponds to the local neighbors, c is a constant term and

$$\rho_1(x) = \frac{\Delta_0^2}{2} \left(\sqrt{1 + \frac{4(x_s - x_r)^2}{\Delta_0^2}} - 1 \right), \Delta_0 > 0. \quad (17)$$

Now, substituting the particular expression (16) for $\log g(x)$ into equation (12), it can be obtained the complete form of the MAP entropy estimator for image segmentation degraded with non-Gaussian noise:

$$\hat{x}_{\text{MAPEE}} = \arg \min_{x \in \mathbb{X}} \left\{ H_A(\hat{p}_{n,h}(e_E)) + \lambda \sum_{\{s,r\} \in \mathbb{C}} b_{sr} \rho_1(x) \right\}. \quad (18)$$

4. EXPERIMENTS AND RESULTS.

In order to evaluate the performance of the new proposed approach of MAP entropy estimation applied to image segmentation, we present a set of experiments with some images, Fig. 2 shows the set of test images used. The first one is an attempt to apply this new approach to medical imaging and the second one is for the case of geographical imaging.

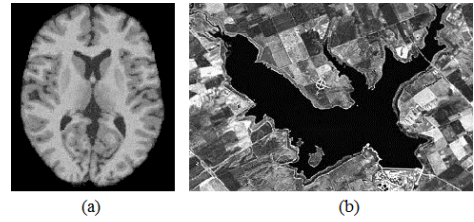


Fig. 2. Set of test images: (a) MR image, (b) geographical image of a dam.

The experiments was performed on a Mac Pro computer with a 2×2.8 GHz quad-core Intel Xeon processor and 2 GB at 800 MHz DDR2 RAM. The minimization process was made using the Levenberg-Marquardt algorithm provided in the optimization toolbox of MATLAB R2009a, where we needed to provide the initial value X_0 to start the search of the solution. The two images were degraded with impulsive noise: `imnoise(X, 'salt&pepper', 0.15)`, and the aim is to obtain the segmentation of the image in spite of the noise present.

For a first experiment we used a generic image of the brain, trying to separate in three tissues: gray matter, white matter and cerebrospinal fluid (CSF). Fig. 3 shows the segmentation results from the noisy image for parameter values $\lambda = 1$ and $\Delta_0 = 110$, with an initial value of $X_0 = 90$. Segmentation results obtained with the new approach of entropy estimation are compared with those obtained applying the segmentation process assuming Gaussian noise for the first term of the MAP estimator, equation (3). Specifically, from [6]:

$$\hat{x}_{\text{MAP}} = \arg \min_{x \in \mathbb{X}} \left\{ \sum_{s \in \mathbb{S}} |y_s - x_s|^2 + \lambda \sum_{\{s,r\} \in \mathbb{C}} b_{sr} \rho_1(x) \right\}. \quad (19)$$

Fig. 3(a) shows the image degraded with impulsive noise, as described in the previous paragraph, Fig. 3(b) shows the segmented image with the MAP entropy estimation approach, equation (18), and Fig. 3(c) shows the segmented image with Gaussian assumption, equation (19). Table I contains information about parameter values and times of computation.

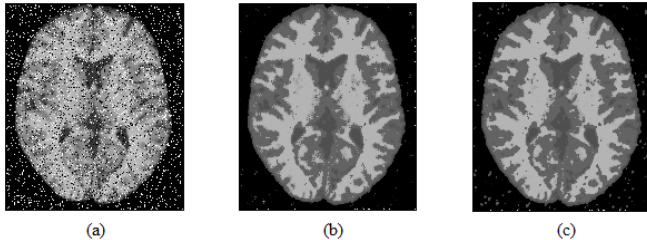


Fig. 3. Segmentation of an image of the brain: (a) image degraded by impulsive noise, (b) segmented image using MAPEE, (c) segmented image using MAP.

TABLE I
NUMERICAL RESULTS FOR THE SEGMENTATION OF BRAIN IMAGE

	X_0	Δ_0	Time (s)
MAPEE	90	110	457.9916
MAP	90	110	333.7739

By using the new approach of entropy estimation, time of computation increases, but in return the error in segmentation result is reduced significantly. In the black background the MAP result presents more and bigger gray spots, and in the part of white matter of the brain it can be seen also the same effect.

A second experiment was made with a geographical image of a dam named Paso de las Piedras, located in Argentina, taken from Google Earth. For this image the interest is on the segmentation of the water region, excluding all other elements. Figure 4 shows segmented images applying both approaches and Table II presents information about the realization of these two processes. As in the previous experiment, it is visually perceptible in the water region that MAPEE improves the segmentation result, for the case of impulsive noise, with respect to the previous approach of Gaussian noise assumption (MAP).

5. CONCLUSIONS.

It was proposed a new approach for image segmentation where it can be considered not only Gaussian noise, but also other kind of degradation factors. In this work it was used impulsive noise that is one of the most degrading and difficult to deal with. It was proved that this new approach produces very good results in the sense of robustness, adapting to the nature of the degradation in the images. We are working in the improvement of this new proposal by adding additional filtering to enhance the final result.

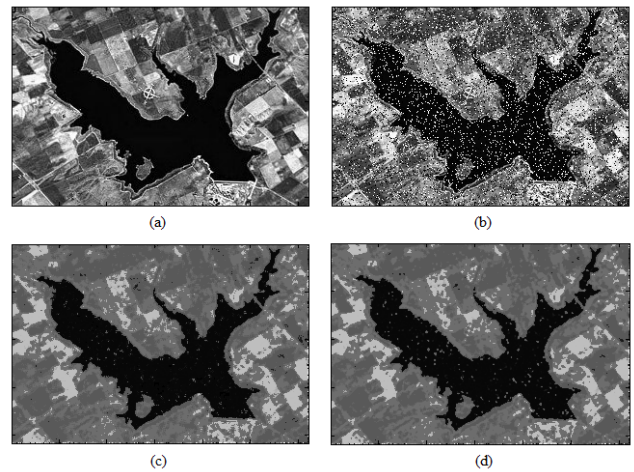


Fig. 5. Segmentation of a geographical image of a dam: (a) original image, (b) image degraded by impulsive noise, (c) segmented image using MAPEE, (d) segmented image using MAP.

TABLE II
NUMERICAL RESULTS FOR THE SEGMENTATION OF GEOGRAPHICAL IMAGE

	X_0	Δ_0	Time (s)
MAPEE	100	130	957.1709
MAP	100	130	725.4942

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